Revision Questions

1. (i) Using tensor notation simplify the following expression:

$$\nabla\left(\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}\right)+\frac{3(\mathbf{A}\cdot\mathbf{r})\,\mathbf{r}}{r^5}.$$

Here $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}, r=|\mathbf{r}|$ and $\mathbf{A}$ is a constant 3D vector field.

(ii) Given the vector field

$$\mathbf{F}(\beta)=(2xyz^3+ye^{xy})\,\mathbf{i}+(x^2z^3+xe^{xy})\,\mathbf{j}+(\beta x^2yz^2+\cos z)\,\mathbf{k},$$

find the value of the constant β (β_0 say) that makes $\mathbf{F}$ irrotational. For this value of β determine the corresponding potential function.

(iii) Find the value of

$$\int_\gamma \mathbf{F}(\beta_0)\cdot\, d\mathbf{r}$$

where γ is the straight line joining the points $(1,2,0)$ and $(2,2,\pi)$ by the following two methods:

(a) use of the potential found in (iii);
(b) direct evaluation using a suitable parameterisation of the path γ.

2. A surface S is given in parametric form by

$$x=u\cos\theta,\ \ y=u\sin\theta,\ \ z=\frac{1}{2}(u^2-1),\ \ (0\le\theta<2\pi,\ \ 0\le u\le 1).$$

(i) Calculate the surface area of S.

(ii) Show that in Cartesian coordinates the surface S is described by

$$z=\frac{1}{2}(x^2+y^2-1),$$

and write down the range of values that z takes. Calculate the volume enclosed by the surface S and the plane $z=0$ by the following two methods:

(a) use of the divergence theorem;
(b) direct evaluation of the appropriate volume integral.

3. (i) Using spherical polar coordinates, evaluate the integral

$$\int_V \frac{z^2\,\mathrm{d}V}{(x^2+y^2+z^2)(1+x^2+y^2+z^2)},$$

where V is the region between concentric spheres centred at the origin with radii a and b $(b > a)$.

(ii) Show that the extremal curve $y = y(x)$ of the integral

$$\int_0^1 y\;dx$$

satisfying the conditions

$$y = 0 \;\text{ when }\; x = 0 \;\text{ and }\; x = 1$$

and the constraint

$$\int_0^1 \left(1 + (dy/dx)^2\right)^{1/2} dx = L$$

is an arc of the circle

$$(x-a)^2 + (y-b)^2 = \lambda^2.$$

Show that $a = 1/2, b^2 = \lambda^2 - 1/4$ and find a relationship between λ and L.
Hence show that if $L = \pi/2$ the circle is centred at $(1/2, 0)$ and has radius $1/2$.

The following integrals may be useful:

$$\int \frac{dx}{\sqrt{\alpha^2 - x^2}} = \sin^{-1}(x/\alpha) + C, \qquad \int \frac{x\;dx}{\sqrt{\alpha^2 - x^2}} = -\sqrt{\alpha^2 - x^2} + C,$$

where C and α are constants.