MVC Revision Questions - Solutions

1. (i) We have $\underline{A}\cdot\underline{r} = A_j x_j$ & $r^2 = x_k x_k$

$$\left[\nabla\left(\frac{\underline{A}\cdot\underline{r}}{r^3}\right)\right]_i = \frac{\partial}{\partial x_i}\left\{A_j x_j (x_k x_k)^{-3/2}\right\}$$

$$= A_j x_j \frac{\partial}{\partial x_i}(x_k x_k)^{-3/2} + (x_k x_k)^{-3/2}\frac{\partial}{\partial x_i}(A_j x_j)$$

$$= A_j x_j \left(-\frac{3}{2}\right)(x_k x_k)^{-5/2} 2x_i + (x_k x_k)^{-3/2} A_i$$

$$= \frac{-3A_j x_j x_i}{(x_k x_k)^{5/2}} + \frac{A_i}{(x_k x_k)^{3/2}}$$

$$\left[\frac{3(\underline{A}\cdot\underline{r})\underline{r}}{r^5}\right]_i = \frac{3A_j x_j x_i}{(x_k x_k)^{5/2}}$$

$$\therefore \left[\nabla\left(\frac{\underline{A}\cdot\underline{r}}{r^3}\right) + \frac{3(\underline{A}\cdot\underline{r})\underline{r}}{r^5}\right]_i = \frac{A_i}{(x_k x_k)^{3/2}}$$

Hence the answer is $\underline{\underline{\underline{A}/r^3}}$

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1 (ii) $\text{Curl } \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 + ye^{xy} & x^2z^3 + xe^{xy} & \beta x^2yz^2 + \cos z \end{vmatrix}$

$$= \hat{i}\left(\beta x^2z^2 - 3x^2z^2\right) - \hat{j}\left(2\beta xyz^2 - 6xyz^2\right) + \hat{k}\left(2xz^3 + e^{xy} + xye^{xy} - 2xz^3 - e^{xy} - xye^{xy}\right)$$

$= 0$ provided $\beta = \beta_0 = 3$

$\underline{F} = \nabla\varphi$

$\Rightarrow \dfrac{\partial\varphi}{\partial x} = F_1 = 2xyz^3 + ye^{xy} \Rightarrow \varphi = x^2yz^3 + e^{xy} + g(y,z)$

Then $\dfrac{\partial\varphi}{\partial y} = x^2z^3 + xe^{xy} + \dfrac{\partial g}{\partial y} = F_2 = x^2z^3 + xe^{xy}$

$\Rightarrow \dfrac{\partial g}{\partial y} = 0$ i.e. $g = g(z)$ only

Then $\dfrac{\partial\varphi}{\partial z} = 3x^2yz^2 + g'(z) = F_3 = \beta_0 x^2yz^2 + \cos z$

$\Rightarrow \beta_0 = 3$ (known already)

& $g(z) = \sin z + C$

$\therefore \varphi = x^2yz^3 + e^{xy} + \sin z + C$

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1 (iii) (a) $\int_\gamma \underline{F}(p_0) \cdot d\underline{r} = \Big[\varphi\Big]_{(1,2,0)}^{(2,2,\pi)}$

$= 2^2 2\pi^3 + e^4 + \sin\pi$

$\quad - 1^2 2 0^3 - e^2 - \sin 0$

$= \underline{\underline{8\pi^3 + e^4 - e^2}}$

(b) Directly : pzn $z = \pi t$, $x = 1+t$, $y = 2$ $(0 \leq t \leq 1)$

$\Rightarrow dx = dt,\ dy = 0,\ dz = \pi\, dt$

$\underline{F} \cdot d\underline{r} = F_1\, dx + F_2\, dy + F_3\, dz$

$= 2(1+t)2\pi^3 t^3 + 2e^{2(1+t)}\, dt$

$+$ zero

$+ (3(1+t)^2 2\pi^2 t^2 + \cos\pi t)\,\pi\, dt$

$\therefore \int_\gamma \underline{F} \cdot d\underline{r} = \int_0^1 4\pi^3(t^3+t^4) + 2e^2 e^{2t} + 6\pi^3(t^4+2t^3+t^2) + \pi\cos\pi t \quad dt$

$= 4\pi^3\left(\frac{1}{4}+\frac{1}{5}\right) + e^2(e^2-1) + 6\pi^3\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{3}\right)$

$+ \sin\pi - \sin 0$

$= \underline{\underline{e^4 - e^2 + 8\pi^3}}$

MVC Revision Questions – Solutions

2/ (i) Calculate Jacobian

$$\underline{J} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & u \\ -u\sin\theta & u\cos\theta & 0 \end{vmatrix}$$

$$= \hat{\imath}(-u^2\cos\theta) - \hat{\jmath}(u^2\sin\theta) + \hat{k}(u\cos^2\theta + u\sin^2\theta)$$

$$= -u^2\cos\theta\,\hat{\imath} - u^2\sin\theta\,\hat{\jmath} + u\hat{k}$$

$$\therefore |\underline{J}| = \sqrt{(u^4\cos^2\theta + u^4\sin^2\theta + u^2)}$$

$$= u\sqrt{(u^2+1)}$$

$$\text{Surface area of } S = \int_S dS = \int_{\theta=0}^{2\pi}\int_{u=0}^{1} |\underline{J}|\, du\, d\theta$$

$$= 2\pi\int_0^1 u(u^2+1)^{1/2}\, du$$

(let $q = u^2$)

$$= 2\pi\int_0^1 \frac{1}{2}(q+1)^{1/2}\, dq = \pi\left[\frac{(q+1)^{3/2}}{3/2}\right]_0^1$$

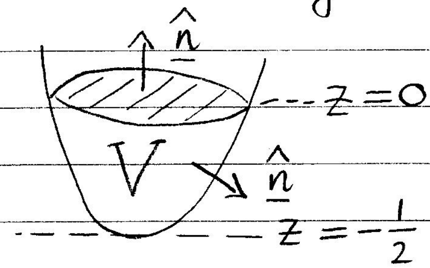
$$= \underline{\underline{\frac{2\pi}{3}\left(2^{3/2}-1\right)}}$$

MVC Revision Questions – Solutions

2/ (ii) $x^2+y^2 = u^2 = 2z+1 \Rightarrow z = \frac{1}{2}(x^2+y^2-1)$

We have $0 \leq u \leq 1 \Rightarrow \underline{\underline{-\frac{1}{2} \leq z \leq 0}}$

(a) Form closed surface by adding circular cap at $z=0$



Call closed surface S'

Divergence theorem : $\oint_{S'} \underline{A} \cdot \hat{\underline{n}}\, dS = \int_V \operatorname{div} \underline{A}\, dV$

for simplicity take $\underline{A} = z\hat{k}$ (other choices possible)

Then: $$\underbrace{\int_S z\hat{\underline{k}} \cdot \hat{\underline{n}}\, dS}_{\text{project onto } z=0} + \underbrace{\int_{cap} z\, \hat{\underline{k}} \cdot \hat{\underline{k}}\, dx\,dy}_{=0 \text{ since } z=0 \text{ on cap}} = \underbrace{\int_V (1)\, dV}_{=V}$$

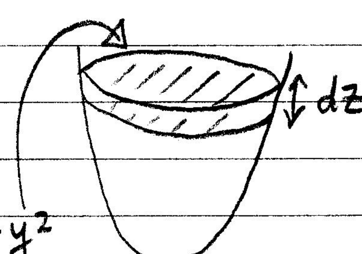
$$\int_{cap} z\hat{\underline{k}} \cdot \hat{\underline{n}} \frac{dx\,dy}{|\hat{\underline{n}} \cdot \hat{\underline{k}}|} = -\int_{\substack{cap \\ x^2+y^2 \leq 1}} \frac{1}{2}(x^2+y^2-1)\, dx\,dy \quad (\text{since } \hat{\underline{n}} \cdot \hat{\underline{k}} < 0)$$

using plane polars

$$= -\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{1}{2}(r^2-1)\, r\, dr\, d\theta = -\pi \left[\frac{r^4}{4} - \frac{r^2}{2}\right]_0^1 = \underline{\underline{\frac{\pi}{4}}}$$

(b) Directly:

$$V = \int_{z=-\frac{1}{2}}^{0} \int_{\theta=0}^{2\pi} \int_{r=0}^{(2z+1)^{1/2}} r\, dr\, d\theta\, dz$$



$x^2+y^2 = 2z+1$

$$= 2\pi \int_{-\frac{1}{2}}^{0} \frac{2z+1}{2}\, dz$$

$$= \pi \left[z^2+z\right]_{-1/2}^{0} = \underline{\underline{\frac{\pi}{4}}}$$

volume element in cylindrical polars

MVC Revision Questions - Solutions

3/ (i) We have $dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$

$$z^2 = r^2 \cos^2\theta$$

$$\& \quad x^2 + y^2 + z^2 = r^2$$

$\therefore$ integral becomes $\displaystyle\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=a}^{r=b} \frac{\cos^2\theta \, r^4 \sin\theta}{r^2(1+r^2)} \, dr \, d\theta \, d\varphi$

$$= 2\pi \left[-\frac{\cos^3\theta}{3} \right]_0^{\pi} \int_a^b \frac{r^2}{1+r^2} \, dr$$

$$= \frac{4\pi}{3} \int_a^b 1 - \frac{1}{1+r^2} \, dr$$

$$= \frac{4\pi}{3} \left[r - \tan^{-1}(r) \right]_a^b$$

$$= \frac{4\pi}{3} \left(b - a - \tan^{-1}(b) + \tan^{-1}(a) \right)$$

MVC Revision Questions - Solutions

3/ (ii) Let $f = y$, $g = (1+y'^2)^{1/2}$
and apply Euler-Lagrange equation to $f+\lambda g$

$$\frac{\partial}{\partial y}\left(y+\lambda(1+y'^2)^{1/2}\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\left(y+\lambda(1+y'^2)^{1/2}\right)\right) = 0$$

$$\Rightarrow 1 - \frac{d}{dx}\left(\lambda y'(1+y'^2)^{-1/2}\right) = 0$$

$$\Rightarrow x - a = \lambda y'(1+y'^2)^{-1/2}$$

$$\Rightarrow (x-a)^2(1+y'^2) = \lambda^2 y'^2$$

$$\Rightarrow y'^2 = (x-a)^2/(\lambda^2-(x-a)^2)$$

[Alternatively use short form of E-L - get to same result]

$$\Rightarrow y - b = \pm\int \frac{x-a}{(\lambda^2-(x-a)^2)^{1/2}}\,dx$$

$$= \mp(\lambda^2-(x-a)^2)^{1/2}$$

$$\Rightarrow \underline{(y-b)^2+(x-a)^2 = \lambda^2}\text{, as required.}$$

Apply end conditions:

$y=0$ at $x=0 \Rightarrow a^2+b^2=\lambda^2$

$y=0$ at $x=1 \Rightarrow (1-a)^2+b^2=\lambda^2$

$\Rightarrow a^2=(1-a)^2$

$\Rightarrow \underline{\underline{a=1/2}}$

and $b^2=\lambda^2-a^2$
$=\lambda^2-1/4$

Integral constraint: $\int_0^1 (1+y'^2)^{1/2}\,dx = L$

$$\Rightarrow \int_0^1\left(1+\frac{(x-a)^2}{\lambda^2-(x-a)^2}\right)^{1/2} dx = L$$

$$\Rightarrow \underbrace{\int_0^1 \frac{\lambda\,dx}{(\lambda^2-(x-a)^2)^{1/2}}}_{\lambda\left[\sin^{-1}\left(\frac{x-a}{\lambda}\right)\right]_0^1} = L$$

$$\Rightarrow \frac{L}{\lambda} = \sin^{-1}\left(\frac{1}{2\lambda}\right) - \sin^{-1}\left(\frac{-1}{2\lambda}\right)$$

$$= 2\sin^{-1}(1/2\lambda)$$

$$\Rightarrow \underline{\underline{L = 2\lambda\sin^{-1}\left(\frac{1}{2\lambda}\right)}}$$

If $L=\frac{\pi}{2}$ we see that $\lambda = 1/2$

and hence $b = \lambda^2 - 1/4 = 0$

$\therefore$ centre is $(a,b) = (1/2, 0)$ & radius $= \lambda = 1/2$