Exercise 1.1. (a) Show that the inner product satisfies the following properties: for all $x, y, z \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

$$\langle x, y\rangle = \langle y, x\rangle\,, \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle\,, \qquad \langle ax, y\rangle = a\,\langle x, y\rangle\,.$$

Solution: These are computations using the definition of the inner product, vector addition and scalar multiplication, and linearity properties of sums.

$$\langle x, y\rangle = \sum_{i=1}^{n} x^i y^i = \sum_{i=1}^{n} y^i x^i = \langle y, x\rangle\,.$$

$$\begin{aligned}\langle x+y, z\rangle &= \sum_{i=1}^{n}(x+y)^i z^i = \sum_{i=1}^{n}(x^i+y^i)z^i = \sum_{i=1}^{n}(x^i z^i + y^i z^i)\\ &= \sum_{i=1}^{n} x^i z^i + \sum_{i=1}^{n} y^i z^i = \langle x, z\rangle + \langle y, z\rangle\,.\end{aligned}$$

$$\langle ax, y\rangle = \sum_{i=1}^{n} a x^i y^i = a\sum_{i=1}^{n} x^i y^i = a\,\langle x, y\rangle\,.$$

(b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$\|x+ty\|^2 = \|x\|^2 + 2t\,\langle x, y\rangle + t^2\,\|y\|^2 \geq 0 \tag{1}$$

Solution: We use the properties of the inner product established above to find:

$$\begin{aligned}\|x+ty\|^2 = \langle x+ty, x+ty\rangle &= \langle x, x+ty\rangle + \langle ty, x+ty\rangle\\ &= \langle x, x\rangle + \langle x, ty\rangle + \langle ty, x\rangle + \langle ty, ty\rangle\\ &= \langle x, x\rangle + 2t\,\langle x, y\rangle + t^2\,\langle y, y\rangle\\ &= \|x\|^2 + 2t\,\langle x, y\rangle + t^2\,\|y\|^2\,.\end{aligned}$$

Since $\|x+ty\|^2 \geq 0$, we certainly have:

$$\|x\|^2 + 2t\,\langle x, y\rangle + t^2\,\|y\|^2 \geq 0.$$

(c) By thinking of (1) as a quadratic in t, and considering its possible roots, deduce the *Cauchy-Schwartz* inequality:

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|\,. \tag{2}$$

When does equality hold?

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Questions marked with $*$ are optional

Solution: (1) is a non-negative quadratic in t, so it can have at most one root. Thus the discriminant ($b^2 - 4ac$ with the usual conventions) must be non-positive, i.e.

$$4\langle x, y\rangle^2 - 4\|x\|^2\|y\|^2 \leq 0,$$

which gives the result on re-arrangement. Equality holds iff there exists $t \in \mathbb{R}$ such that $\|x + ty\| = 0$, which is the condition that x, y are parallel.

(d) Deduce the triangle inequality for the norm on $\mathbb{R}^n$.

Solution: Returning to (1) and setting $t = 1$, we have:

$$\begin{aligned}\|x+y\|^2 &= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2\end{aligned}$$

Since both sides are positive, we deduce that:

$$\|x + y\| \leq \|x\| + \|y\|.$$

(e) Show the reverse triangle inequality:

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\|$$

Solution: To see the above inequality, it is enough to show that

$$\|x\| - \|y\| \leq \|x - y\| \quad \text{and} \quad \|x\| - \|y\| \geq -\|x - y\|.$$

For the first one, we note that

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|$$

which gives is the first inequality. For the second one, we note that

$$\|y\| = \|(y - x) + x\| \leq \|x - y\| + \|x\|$$

which gives the second inequality by rearranging the terms.

Exercise 1.2. Suppose $x = (x^1, \dots, x^n) \in \mathbb{R}^n$.

(i) Show that:

$$\max_{k=1,\dots,n} \left|x^k\right| \leq \|x\|.$$

Solution: Fix an arbitrary k in $\{1, 2, \dots, n\}$. Since $y \mapsto \sqrt{y}$ is an increasing map from $[0, +\infty)$ to $[0, +\infty)$, we have

$$\left|x^k\right| = \sqrt{(x^k)^2} \leq \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2} = \|x\|.$$

This implies that the maximum of all these numbers is bounded by $\|x\|$.

(ii) Show that:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|x^k\right|.$$

[Hint: write out $\|x\|^2$ in coordinates and estimate]

Solution: Writing out $\|x\|^2$, we have:

$$\|x\|^2 = \sum_{i=1}^{n} (x^i)^2 \leq \sum_{i=1}^{n} \max_{k=1,\ldots,n} (x^k)^2 = \sum_{i=1}^{n} \left(\max_{k=1,\ldots,n} \left|x^k\right| \right)^2 = n \left(\max_{k=1,\ldots,n} \left|x^k\right| \right)^2.$$

Taking square roots, we have:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|x^k\right|,$$

since both sides are positive.

Exercise 1.3. Suppose that $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ are two sequences in $\mathbb{R}^n$ with

$$\lim_{i\to\infty} x_i = x, \qquad \lim_{i\to\infty} y_i = y.$$

(a) Show that

$$\lim_{i\to\infty} (x_i + y_i) = x + y.$$

Solution: Fix $\epsilon > 0$. By the convergence of (x_i), (y_i) there exists N_1, N_2 such that for $i \geq N_1$ and $j \geq N_2$ we have:

$$\|x_i - x\| < \frac{\epsilon}{2}, \qquad \|y_j - y\| < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$. Then if $i \geq N$ we have:

$$\|x_i + y_i - (x + y)\| \leq \|x_i - x\| + \|y_i - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) Show that

$$\lim_{i\to\infty} \langle x_i, y_i \rangle = \langle x, y \rangle,$$

and deduce that

$$\lim_{i\to\infty} \|x_i\| = \|x\|.$$

[Hint: Write $\langle x_i, y_i \rangle - \langle x, y \rangle = \langle x_i - x, y_i - y \rangle + \langle x_i - x, y \rangle + \langle x, y_i - y \rangle$ and use the Cauchy-Schwartz inequality (2)]

Solution: Fix $\epsilon > 0$, and without loss of generality assume $\epsilon < 1$. By the convergence of (x_i), (y_i) there exists N_1, N_2 such that for $i \geq N_1$ and $j \geq N_2$ we have:

$$\|x_i - x\| < \frac{\epsilon}{3(1 + \|y\|)}, \qquad \|y_j - y\| < \frac{\epsilon}{3(1 + \|x\|)}.$$

(The reason for the above choices will be clear in a moment.)

Set $N = \max\{N_1, N_2\}$. Then, for all $i \geq N$, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\langle x_i, y_i\rangle - \langle x, y\rangle| &= |\langle x_i, y_i\rangle - \langle x_i, y\rangle + \langle x_i, y\rangle - \langle x, y\rangle| \\
&= |\langle x_i, y_i - y\rangle + \langle x_i - x, y\rangle| \\
&\leq |\langle x_i, y_i - y\rangle| + |\langle x_i - x, y\rangle| \\
&\leq \|x_i\| \, \|y_i - y\| + \|x_i - x\| \, \|y\| \\
&= \|x_i - x + x\| \, \|y_i - y\| + \|x_i - x\| \, \|y\| \\
&\leq (\|x_i - x\| + \|x\|) \, \|y_i - y\| + \|x_i - x\| \, \|y\| \\
&\leq \|x_i - x\| \, \|y_i - y\| + \|x\| \, \|y_i - y\| + \|x_i - x\| \, \|y\| \\
&< \frac{\epsilon^2}{9(1 + \|y\|)(1 + \|x\|)} + \|x\| \, \frac{\epsilon}{3(1 + \|x\|)} + \|y\| \, \frac{\epsilon}{3(1 + \|y\|)} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

(c) Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence of real numbers with $a_i \to a$ as $i \to \infty$. Show that

$$\lim_{i \to \infty} (a_i x_i) = ax.$$

[Hint: Write $a_i x_i - ax = (a_i - a)(x_i - x) + (a_i - a)x + a(x_i - x)$ and use the properties of the norm.]

Solution: Fix $\epsilon > 0$, and without loss of generality assume $\epsilon < 1$. By the convergence of (a_i), (y_i) there exists N_1, N_2 such that for $i \geq N_1$ and $j \geq N_2$ we have:

$$\|x_i - x\| < \frac{\epsilon}{3(1 + |a|)}, \qquad |a_j - a| < \frac{\epsilon}{3(1 + \|x\|)}.$$

Set $N = \max\{N_1, N_2\}$. Then if $i \geq N$ we have:

$$\begin{aligned}
\|a_i x_i - ax\| &= \|(a_i - a)(x_i - x) + (a_i - a)x + a(x_i - x)\| \\
&\leq \|(a_i - a)(x_i - x)\| + \|(a_i - a)x\| + \|a(x_i - x)\| \\
&= |a_i - a| \, \|x_i - x\| + |a_i - a| \, \|x\| + |a| \, \|x_i - x\| \\
&< \frac{\epsilon^2}{9(1 + |a|)(1 + \|x\|)} + \epsilon \frac{\|x\|}{3(1 + \|x\|)} + \epsilon \frac{|a|}{3(1 + |a|\,)} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Exercise 1.4. Which of the following subsets of $\mathbb{R}^n$ is open:

(a) $\mathbb{R}^n$?

(b) $\emptyset$?

(c) $\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 > 0\}$?

(d) $\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : x^i \in [0, 1)\}$?

(e) $\mathbb{Q}^n := \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : x^i \in \mathbb{Q}\}$?

Solution: a) Open, b) Open, c) Open, d) Not open, e) Not Open.

Exercise 1.5. Let $(x_i)_{i=0}^{\infty}$ be a sequence of vectors $x_i \in \mathbb{R}^n$ with $x_i \to x$. Suppose that the x_i satisfy $\|x_i\| < r$ for all i and some $r > 0$. Show that:

$$\|x\| \leq r.$$

[Hint: work by contradiction, assume $\|x\| > r$ and show this leads to an absurdity]

Solution: Suppose in the contrary that $\|x\| = s > r$. Let $\epsilon = \frac{s-r}{2} > 0$. By the convergence of (x_i), there exists $j \in \mathbb{N}$ such that:

$$\|x_j - x\| < \epsilon.$$

By the reverse triangle inequality we have:

$$\big|\|x\| - \|x_j\|\big| \leq \|x_j - x\| < \epsilon,$$

however:

$$\big|\|x\| - \|x_j\|\big| = s - \|x_j\| \geq s - r = 2\epsilon,$$

so we conclude

$$2\epsilon < \epsilon$$

which together with the fact that $\epsilon > 0$ is a contradiction.

Exercise 1.6. (a) Show that if U_1, U_2 are open in $\mathbb{R}^n$, then so are the sets

$$i)\ \ U_1 \cup U_2 \qquad\qquad ii)\ \ U_1 \cap U_2$$

Solution: Suppose $x \in U_1 \cup U_2$. Then either $x \in U_1$ or $x \in U_2$. WLOG consider the first possibility. Then since U_1 is open, there exists $r > 0$ such that $B_r(x) \subset U_1$. But this implies $B_r(x) \subset U_1 \cup U_2$, so $U_1 \cup U_2$ is open.

Suppose $x \in U_1 \cap U_2$. Then there exist r_1, r_2 such that $B_{r_1}(x) \subset U_1$ and $B_{r_2}(x) \subset U_2$. Taking $r = \min\{r_1, r_2\}$ we have:

$$B_r(x) \subset B_{r_1}(x) \subset U_1, \qquad B_r(x) \subset B_{r_2}(x) \subset U_2,$$

so that $B_r(x) \subset U_1 \cap U_2$ and thus $U_1 \cap U_2$ is open.

(b) Suppose U_α, for α in an index set I, is a collection of open sets in $\mathbb{R}^n$.

(i) Show that $\bigcup_{\alpha \in I} U_\alpha$ is open in $\mathbb{R}^n$.

Solution: Suppose $x \in \bigcup_{\alpha \in I} U_\alpha$. Then there exists $a \in I$ such that $x \in U_a$. Since U_a is open, there exists $r > 0$ such that $B_r(x) \subset U_a$, which implies $B_r(x) \subset \bigcup_{\alpha \in I} U_\alpha$, hence $\bigcup_{\alpha \in I} U_\alpha$ is open.

(ii) Give an example showing that $\bigcap_{\alpha \in I} U_\alpha$ need not be open.

Solution: Consider:

$$U_i = \left(-2^{-i}, 2^{-i}\right), \text{ for } i \in \mathbb{N}.$$

Then, $\bigcap_{i \in \mathbb{N}} U_i = \{0\}$, which is not open, but each set U_i is an open interval.

Exercise 1.7. Suppose $A \subset \mathbb{R}^n$ is an open set and $f : A \to \mathbb{R}^m$. Show that $\lim_{x\to p} f(x) = F$ if and only if for any sequence $(x_i)_{i=0}^{\infty}$ in $A \setminus \{p\}$ which converges to p we have

$$f(x_i) \to F, \qquad \text{as } i \to \infty.$$

Solution: First suppose that $\lim_{x\to p} f(x) = F$. Then given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in A$ with $0 < \|x - p\| < \delta$ we have:

$$\|f(x) - F\| < \epsilon.$$

Now let $(x_i)_{i=0}^{\infty}$ be any sequence with $x_i \in A, x_i \neq p$ and $x_i \to p$. Since $x_i \to p$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have:

$$0 < \|x_i - p\| < \delta,$$

so by our assumption we have

$$\|f(x_i) - F\| < \epsilon,$$

and thus $f(x_i) \to F$.

Now suppose that for any sequence $(x_i)_{i=0}^{\infty}$ with $x_i \in A, x_i \neq p$ and $x_i \to p$ we have:

$$f(x_i) \to F, \qquad \text{as } i \to \infty.$$

Suppose that $f(x) \not\to F$ as $x \to p$. Then there exists $\epsilon > 0$ such that for any $i \in \mathbb{N}$ we can find x_i with:

$$0 < \|x_i - p\| < 2^{-i}, \qquad \|f(x_i) - F\| \geq \epsilon.$$

Now, clearly the sequence $(x_i)_{i=0}^{\infty}$ converges to p, but $f(x_i) \not\to F$, so we have a contradiction.

Exercise 1.8. (a) Show that the map $f : \mathbb{R} \to \mathbb{R}^n$ defined as $f(x) = (x, 0, \ldots, 0)$ is continuous on $\mathbb{R}$.

Solution: Suppose $p \in \mathbb{R}$. Fix $\epsilon > 0$ and suppose $x \in \mathbb{R}$ satisfies $|x - p| < \epsilon$. Then:

$$\|f(x) - f(p)\| = \|(x - p, 0, \ldots, 0)\| = |x - p| < \epsilon.$$

(b) Let $A \subset \mathbb{R}^n$ and suppose we are given a map $f : A \to \mathbb{R}^m$ where

$$f(x^1, \ldots, x^n) \mapsto \left(f^1\left((x^1, \ldots, x^n)\right), \ldots, f^m\left((x^1, \ldots, x^n)\right)\right).$$

Show that f is continuous at $p \in A$ if and only if each map $f^k : A \to \mathbb{R}$ is continuous at p, for $k = 1, \ldots, m$.

Solution: First suppose that each map $f^k : \mathbb{R}^n \to \mathbb{R}$ is continuous at p, for $k = 1, \ldots, m$. Fix $\epsilon > 0$. Then for each k there exists $\delta_k > 0$ such that for $x \in A$ with $\|x - p\| < \delta_k$ we have:

$$\left|f^k(x) - f^k(p)\right| < \frac{\epsilon}{\sqrt{n}}.$$

Let $\delta = \min_{k=1,\ldots,m} \delta_k$. If $x \in A$, $\|x - p\| < \delta$, we have:

$$\|f(x) - f(p)\| \leq \sqrt{n} \max_{k=1,\ldots,m} \left|f^k(x) - f^k(p)\right| < \sqrt{n}\frac{\epsilon}{\sqrt{n}} = \epsilon,$$

so that f is continuous at p.

Now suppose that f is continuous at p. Fix $\epsilon > 0$, then there exists $\delta > 0$ such that for all $x \in A$, $0 < \|x - p\| < \delta$ we have:

$$\|f(x) - f(p)\| < \epsilon.$$

Fix $j \in \{1, \dots, m\}$. We estimate:

$$\left|f^j(x) - f^j(p)\right| \leq \max_{k=1,\dots,m} \left|f^k(x) - f^k(p)\right| \leq \|f(x) - f(p)\| < \epsilon,$$

so that f^j is continuous at p.

(c) Show that the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f\big((x^1, x^2, \dots, x^n)\big) = 3x^1(x^2)^5 + 4x^2(x^n)^7$ is continuous on $\mathbb{R}^n$, [1].

Solution: By part a), the map from $\mathbb{R}^n$ to each coordinate is continuous, so any finite combination of sums and products of these functions is continuous.

Exercise 1.9.*

(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$, and suppose $U \subset \mathbb{R}^m$ is open. Show that:

$$f^{-1}(U) := \{x \in \mathbb{R}^n : f(x) \in U\}$$

is open.

Solution: Fix $x \in f^{-1}(U)$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$. Since f is continuous, there exists $\delta > 0$ such that if $y \in \mathbb{R}^n$ with $\|y - x\| < \delta$ then $\|f(y) - f(x)\| < \epsilon$. But this implies that $f(y) \in B_\epsilon(f(x)) \subset U$, so we have that $y \in f^{-1}(U)$ provided $\|y - x\| < \delta$. Thus $B_\delta(x) \subset f^{-1}(U)$ and $f^{-1}(U)$ is indeed open.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U) \subset \mathbb{R}^n$ is open for every open $U \subset \mathbb{R}^m$. Show that f is continuous on $\mathbb{R}^n$.

Solution: Fix $x \in \mathbb{R}^n$, and let $\epsilon > 0$. Since $B_\epsilon(f(x))$ is open, we have that the set $f^{-1}(B_\epsilon(f(x)))$ is open. We note that $x \in f^{-1}(B_\epsilon(f(x)))$, thus there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Now if $y \in \mathbb{R}^n$ with $\|x - y\| < \delta$, then $y \in B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$, so that $f(y) \in B_\epsilon(f(x))$ and thus $\|f(y) - f(x)\| < \epsilon$, so that f is indeed continuous at x.

Unseen Exercise. Let $\alpha \in \mathbb{R}$ be an irrational number, and for $n \in \mathbb{N}$ let

$$a_n = \frac{1}{2^n}\left(\cos(2\pi n\alpha), \sin(2\pi n\alpha)\right) \in \mathbb{R}^2.$$

(a) Show that $a_n \to (0, 0) \in \mathbb{R}^2$ as $n \to \infty$.

Solution: Let $\epsilon > 0$ be arbitrary. There is $n' \geq 1$ such that for all $n \geq n'$ we have $2^{-n} < \epsilon$. For $n \geq n'$ we have

$$\|a_n - (0,0)\| = |2^{-n}|\, \|(\cos(2\pi n\alpha), \sin(2\pi n\alpha))\| = 2^{-n} < \epsilon.$$

[1] Here, $(x^j)^m$ denotes the coordinate x^j raised to power m.

(b) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ according to

$$f(x) = \begin{cases} 1 & \text{if } x = a_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the map f is not continuous at $(0, 0)$.

Solution: Since $a_n \neq (0, 0)$ for all $n \in \mathbb{N}$, we have $f(0, 0) = 0$. On the other hand $a_n \to (0, 0)$ and $f(a_n) \equiv 1$ does not converge to $0 = f(0, 0)$. This shows that the map f is not continuous at $(0, 0)$.

(c) for every non-zero vector $u = (u^1, u^2) \in \mathbb{R}^2$, show that f is continuous in the direction of u at 0. That is, the map $t \mapsto f(tu)$ is continuous at $t = 0$.

Solution: Let us fix an arbitrary non-zero vector $u = (u^1, u^2) \in \mathbb{R}^2$. Consider the line

$$L = \{tu \mid t \in \mathbb{R}\} \subset \mathbb{R}^2.$$

We claim that there is at most one integer $n \in \mathbb{N}$ such that $a_n \in L$. Assume in the contrary that there are two such integers, say m and n with $m \neq n$. Then, there are t_n and t_m in $\mathbb{R}$ such that $a_m = t_m u$ and $a_n = t_n u$. Because a_n and a_m are non-zero, t_n and t_m must be non-zero, so we conclude that

$$u = a_m/t_m = a_n/t_n,$$

and then

$$\frac{1}{2^m t_m} (\cos(2\pi m\alpha), \sin(2\pi m\alpha)) = \frac{1}{2^n t_n} (\cos(2\pi n\alpha), \sin(2\pi n\alpha)) .$$

Since for every $\gamma \in \mathbb{R}$, $(\cos(\gamma), \sin(\gamma))$ has modulus 1, we conclude that $|2^n t_n| = |2^m t_m|$. Therefore, either

$$(\cos(2\pi m\alpha), \sin(2\pi m\alpha)) = (\cos(2\pi n\alpha), \sin(2\pi n\alpha))$$

or

$$(\cos(2\pi m\alpha), \sin(2\pi m\alpha)) = -(\cos(2\pi n\alpha), \sin(2\pi n\alpha)) .$$

Both of these cases imply that $\cos(2\pi m\alpha) = \cos(2\pi n\alpha)$. This implies that there is $k \in \mathbb{Z}$ such that $2\pi n\alpha = 2\pi m\alpha + 2k\pi$. Therefore, $\alpha = k/(n - m)$, which contradicts α being irrational.

Let us define δ as follows. If there is no a_n in L, we define $\delta = \|u\|$. If there is $a_n \in L$, we let $\delta = \|a_n\| / \|u\|$. Since there is at most one a_n in L, this is a well-defined number.

We claim that for every $t \in \mathbb{R}$ such that $|t| < \delta$, we have $f(tu) = 0$. That is because if there is no a_n in L then $f(tu)$ is constant 0 for every t. If there is $a_n \in L$, then we have

$$\|tu\| < |t| \|u\| < \delta \|\|_u = \|a_n\| .$$

This implies that $f(tu) = 0$.

Since the map $t \mapsto f(tu)$ is constant on the interval $(-\delta, \delta)$, it is continuous at 0.