Problem Sheet 1 Analysis II
Davoud Cheraghi Autumn 2021

Exercise 1.1. (a) Show that the inner product satisfies the following properties: for all $x, y, z \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

$$\langle x, y \rangle = \langle y, x \rangle \,, \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \,, \qquad \langle ax, y \rangle = a \langle x, y \rangle \,.$$

(b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$\|x + ty\|^2 = \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0 \tag{1}$$

(c) By thinking of (1) as a quadratic in t, and considering its possible roots, deduce the *Cauchy-Schwartz* inequality:

$$|\langle x, y \rangle| \leq \|x\| \|y\| \,. \tag{2}$$

When does equality hold?

(d) Deduce the triangle inequality for the norm on $\mathbb{R}^n$.

(e) Show the reverse triangle inequality:

$$\big| \|x\| - \|y\| \big| \leq \|x - y\|$$

Exercise 1.2. Suppose $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.

(i) Show that:

$$\max_{k=1,\ldots,n} \left| x^k \right| \leq \|x\| \,.$$

(ii) Show that:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left| x^k \right| .$$

Exercise 1.3. Suppose that $(x_i)_{i=0}^{\infty}$ and $(y_i)_{i=0}^{\infty}$ are two sequences in $\mathbb{R}^n$ with

$$\lim_{i \to \infty} x_i = x, \qquad \lim_{i \to \infty} y_i = y.$$

(a) Show that

$$\lim_{i \to \infty} (x_i + y_i) = x + y.$$

(b) Show that

$$\lim_{i \to \infty} \langle x_i, y_i \rangle = \langle x, y \rangle \,,$$

and deduce that

$$\lim_{i \to \infty} \|x_i\| = \|x\| \,.$$

(c) Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence of real numbers with $a_i \to a$ as $i \to \infty$. Show that

$$\lim_{i \to \infty} (a_i x_i) = ax.$$

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

Exercise 1.4. Which of the following subsets of $\mathbb{R}^n$ is open:

(a) $\mathbb{R}^n$?

(b) $\emptyset$?

(c) $\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 > 0\}$?

(d) $\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : x^i \in [0, 1)\}$?

(e) $\mathbb{Q}^n := \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : x^i \in \mathbb{Q}\}$?

Exercise 1.5. Let $(x_i)_{i=0}^{\infty}$ be a sequence of vectors $x_i \in \mathbb{R}^n$ with $x_i \to x$. Suppose that the x_i satisfy $\|x_i\| < r$ for all i and some $r > 0$. Show that:

$$\|x\| \leq r.$$

Exercise 1.6. (a) Show that if U_1, U_2 are open in $\mathbb{R}^n$, then so are the sets

$$i)\ U_1 \cup U_2 \qquad\qquad ii)\ U_1 \cap U_2$$

(b) Suppose U_α, for α in an index set I, is a collection of open sets in $\mathbb{R}^n$.

(i) Show that $\bigcup_{\alpha \in I} U_\alpha$ is open in $\mathbb{R}^n$.
(ii) Give an example showing that $\bigcap_{\alpha \in I} U_\alpha$ need not be open.

Exercise 1.7. Suppose $A \subset \mathbb{R}^n$ is an open set and $f : A \to \mathbb{R}^m$. Show that $\lim_{x \to p} f(x) = F$ if and only if for any sequence $(x_i)_{i=0}^{\infty}$ in $A \setminus \{p\}$ which converges to p we have

$$f(x_i) \to F, \quad \text{as } i \to \infty.$$

Exercise 1.8. (a) Show that the map $f : \mathbb{R} \to \mathbb{R}^n$ defined as $f(x) = (x, 0, \ldots, 0)$ is continuous on $\mathbb{R}$.

(b) Let $A \subset \mathbb{R}^n$ and suppose we are given a map $f : A \to \mathbb{R}^m$ where

$$f(x^1, \ldots, x^n) \mapsto \left(f^1\big((x^1, \ldots, x^n)\big), \ldots, f^m\big((x^1, \ldots, x^n)\big)\right).$$

Show that f is continuous at $p \in A$ if and only if each map $f^k : A \to \mathbb{R}$ is continuous at p, for $k = 1, \ldots, m$.

(c) Show that the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f\big((x^1, x^2, \ldots, x^n)\big) = 3x^1(x^2)^5 + 4x^2(x^n)^7$ is continuous on $\mathbb{R}^n$, [1].

Exercise 1.9.*

(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$, and suppose $U \subset \mathbb{R}^m$ is open. Show that:

$$f^{-1}(U) := \{x \in \mathbb{R}^n : f(x) \in U\}$$

is open.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U) \subset \mathbb{R}^n$ is open for every open $U \subset \mathbb{R}^m$. Show that f is continuous on $\mathbb{R}^n$.

---

[1] Here, $(x^j)^m$ denotes the coordinate x^j raised to power m.