Exercise 10.1. Show that any convergent sequence in a metric space, is a Cauchy sequence.

Hint: Adapt the proof of the same statement for the sequences of real numbers.

Solution: Let $(x_n)_{n\geq 1}$ be a sequence in X which converges to $x \in X$. Fix an arbitrary $\epsilon > 0$. By the definition of convergence of sequences, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mathrm{d}(x_n, x) < \epsilon/2$. Therefore, by the triangle inequality, for all $m, n \geq N$ we obtain

$$\mathrm{d}(x_n, x_m) \leq \mathrm{d}(x_n, x) + \mathrm{d}(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $(x_n)_{n\geq 1}$ is Cauchy.

Exercise 10.2. Let (X, d) be a metric space, and assume that $(x_n)_{n\geq 1}$ is a Cauchy sequence in X. If there is a subsequence of $(x_n)_{n\geq 1}$ which converges to some $x \in X$, then the sequence $(x_n)_{n\geq 1}$ converges to x.

Hint: Adapt the proof of the same statement for the sequences of real numbers.

Solution: Let $(x_{n_k})_{k\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$ which converges to some $x \in X$. Fix $\epsilon > 0$. There is $N \in \mathbb{N}$ such that if $k \geq N$ we have $\mathrm{d}(x_{n_k}, x) < \epsilon/2$. On the other hand, there is $M \in \mathbb{N}$ such that if $m, n \geq M$, $\mathrm{d}(x_m, x_n) < \epsilon/2$. For every $n \geq M$, we may choose $k \geq N$ such that $n_k \geq M$. Then, by the triangle inequality,

$$\mathrm{d}(x_n, x) \leq \mathrm{d}(x_n, x_{n_k}) + \mathrm{d}(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $(x_n)_{n\geq 1}$ converges to x.

Exercise 10.3. Let $\mathcal{C}$ be a collection of functions $f : [a, b] \to \mathbb{R}$. Assume that there is $K > 0$ such that for all $f \in \mathcal{C}$ and all x and y in $[a, b]$, we have

$$|f(x) - f(y)| \leq K|x - y|.$$

Show that the family $\mathcal{C}$ is uniformly equi-continuous.

Hint: Show that for ϵ one can use $\delta = \epsilon/K$.

Solution: Fix $\epsilon > 0$. Let $\delta = \epsilon/K$. For all $f \in \mathcal{C}$, and all x and y in $[a, b]$, if $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq K|x - y| < K\delta = \epsilon.$$

This means that the family $\mathcal{C}$ is uniformly equi-continuous.

Exercise 10.4. Let $x_1 = \sqrt{2}$, and define the sequence $(x_n)_{n\geq 1}$ according to

$$x_{n+1} = \sqrt{2 + \sqrt{x_n}}.$$

Show that the sequence $(x_n)_{n\geq 1}$ converges to a root of the equation

$$x^4 - 4x^2 - x + 4 = 0$$

which lies in the interval $[\sqrt{3}, 2]$.

Hint: Work with the function $f(x) = \sqrt{2 + \sqrt{x}}$ on the interval $[\sqrt{3}, 2]$.

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Questions marked with $*$ are optional

Solution: Consider the map
$$f(x) = \sqrt{2+\sqrt{x}}, \quad \forall x \in [\sqrt{3}, 2].$$
First we note that f maps $[\sqrt{3}, 2]$ into $[\sqrt{3}, 2]$. That is because, $f(\sqrt{3}) \geq \sqrt{3}$, $f(2) \leq 2$, and f is an increasing function. More precisely, for all $t \in [\sqrt{3}, 2]$ we have
$$\sqrt{3} \leq f(\sqrt{3}) \leq f(t) \leq f(2) \leq 2,$$
and hence $f(t) \in [\sqrt{3}, 2]$.

Now we show that f is contracting on the interval $[\sqrt{3}, 2]$. By a simple calculation we see that for all $x \in [\sqrt{3}, 2]$, we have
$$f'(x) = \frac{1}{4} \frac{1}{\sqrt{2+\sqrt{x}}} \frac{1}{\sqrt{x}} \leq \frac{1}{4}.$$
Therefore, for all x and y in $[\sqrt{3}, 2]$, we have
$$|f(x) - f(y)| = \left| \int_x^y f'(t)\, dt \right| \leq \left| \int_x^y |f'(t)| dt \right| \leq \frac{1}{4}|x-y|.$$
This shows that f is uniformly contracting on $[\sqrt{3}, 2]$.

By definition, $x_{n+1} = f(x_n)$, the sequence $(x_n)_{n\geq 1}$ is contained in $[\sqrt{3}, 2]$. Since $(\mathbb{R}, \mathrm{d}_1)$ is a complete metric space, and $[\sqrt{3}, 2]$ is closed in $(\mathbb{R}, \mathrm{d}_1)$, we conclude that $[\sqrt{3}, 2]$ is a complete metric space with respect to the induced metric. By the argument in the proof of the Banach fixed point theorem, the sequence $(x_n)_{n\geq 1}$ is a Cauchy sequence. Therefore, it must converge to some limit in $[\sqrt{3}, 2]$. Moreover, the limit of the sequence, is the unique fixed point of the function f in $[\sqrt{3}, 2]$. Thus, we must have
$$x = \sqrt{2+\sqrt{x}}$$
which implies that $x^2 = 2 + \sqrt{x}$, and hence $(x^2-2)^2 = x$, and hence the relation in the exercise.

Exercise 10.5. Consider the map $f : (0, 1/3) \to (0, 1/3)$, defined as $f(x) = x^2$. Show that the map f is a contraction with respect to the Euclidean metric d_1. But, f has no fixed point in $(0, 1/3)$.

Hint: you may use the formula $x^2 - y^2 = (x-y)(x+y)$.

Solution: For all x and y in $(0, 1/3)$, we have
$$|f(x) - f(y)| = |(x-y)(x+y)| < \frac{2}{3}|x-y|$$
Thus, f us contracting. However, since for all $x \in (0, 1/3)$, $f(x) < x$, f does not have a fixed point in $(0, 1/3)$. One cannot apply the Banach Fixed point theorem here since the interval $(0, 1/3)$ is not complete.

Exercise 10.6. Consider the map $f : [1, \infty) \to [1, \infty)$ defined as $f(x) = x + 1/x$. Show that $([1, +\infty), \mathrm{d}_1)$ is a complete metric space, and for all x and y in $[1, \infty)$ we have
$$\mathrm{d}_1(f(x), f(y)) \leq \mathrm{d}(x, y).$$
But, f has no fixed point.

Hint: You may use $f' < 1$ on $[1, +\infty)$.

Solution: Any Cauchy sequence in $([1,+\infty), \mathrm{d}_1)$ is a Cauchy sequence in $(\mathbb{R}, \mathrm{d}_1)$. But the latter space is complete so that the sequence converges to some $x \in (\mathbb{R}, \mathrm{d}_1)$. This x also belongs to $[1,+\infty)$ since this set is closed. Thus $([1,+\infty), \mathrm{d}_1)$ is complete.

We note that for all x and y in $[1,+\infty)$, we have

$$\begin{aligned} |f(x) - f(y)| &= \left| x - y + \frac{1}{x} - \frac{1}{y} \right| \\ &= \left| x - y - \frac{x-y}{xy} \right| \\ &\leq |x-y| \cdot \left| 1 - \frac{1}{xy} \right| \\ &\leq |x-y|. \end{aligned}$$

Obviously, $f(x)$ has no fixed point, since for all x in $[1,+\infty$ we have $x \neq x + 1/x$. Note that the Banach Fixed Point Theorem cannot be applied here since there is no $K \in (0,1)$ such that $|f(x) - f(y)| \leq K|x-y|$.

Unseen Exercise. (unseen) Let $\mathcal{C}$ be a collection of functions $f : [a,b] \to \mathbb{R}$. Assume that there are $K > 0$ and $\alpha > 0$ such that for all $f \in \mathcal{C}$ and all x and y in $[a,b]$, we have

$$|f(x) - f(y)| \leq K|x-y|^\alpha.$$

Show that the family $\mathcal{C}$ is uniformly equi-continuous. A function f satisfying this inequality for some K and α, is called a holder function (or an α-holder function).

Hint: Show that for ϵ one can use $\delta = (\epsilon/K)^{1/\alpha}$.

Solution: Fix $\epsilon > 0$. Let $\delta = (\epsilon/K)^{1/\alpha}$. For all $f \in \mathcal{C}$, and all x and y in $[a,b]$, if $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq K|x-y|^\alpha < K\delta^\alpha = \epsilon.$$

This means that the family $\mathcal{C}$ is uniformly equi-continuous.