Exercise 2.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f(x) = x.$$

Show that f is differentiable at each $p \in \mathbb{R}^n$ and

$$Df(p) = \text{id},$$

where $\text{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

Solution: We could appeal to an example in the lecture notes (linear maps are differentiable) and note that the identity is a linear map, thus is differentiable with derivative equal to itself. Alternatively, we note that if $Df(p) = \iota$, then

$$f(p + h) - f(p) - Df(p)[h] = (p + h) - p - h = 0,$$

so we certainly have

$$\lim_{h\to 0} \frac{\|f(p + h) - f(p) - Df(p)[h]\|}{\|h\|} = 0,$$

which implies f is differentiable.

Exercise 2.2. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) \mapsto x^2 + y^2,$$

is differentiable at all points $p = (\xi, \eta) \in \mathbb{R}^2$ with Jacobian

$$Df(p) = (2\xi \quad 2\eta)$$

Solution: Setting $h = (h_1, h_2)$, we calculate

$$\begin{aligned} f(p + h) - f(p) - Df(p)[h] &= (\xi + h_1)^2 + (\eta + h_2)^2 - \xi^2 - \eta^2 - 2\xi h_1 - 2\eta h_2 \\ &= h_1^2 + h_2^2. \end{aligned}$$

Thus we have

$$\frac{\|f(p + h) - f(p) - Df(p)[h]\|}{\|h\|} = \frac{h_1^2 + h_2^2}{\sqrt{h_1^2 + h_2^2}} = \sqrt{h_1^2 + h_2^2},$$

so certainly

$$\lim_{h\to 0} \frac{\|f(p + h) - f(p) - Df(p)[h]\|}{\|h\|} = \lim_{h\to 0} \|h\| = 0.$$

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Questions marked with $*$ are optional

Exercise 2.3. One might hope that the differential can be calculated by finding

$$\lim_{x\to p} \frac{f(x)-f(p)}{\|x-p\|}.$$

By considering the example of Exercise 2.1 or otherwise, show that this limit may not always exist, even if f is differentiable at p.

Solution: Taking $f:\mathbb{R}^n\to\mathbb{R}^n$ to be the identity and $p=0$, we have

$$\frac{f(x)-f(p)}{\|x-p\|} = \frac{x}{\|x\|}.$$

This function has no limit as $x\to 0$. To see this, consider $x=\lambda e_1$, then:

$$\frac{x}{\|x\|} = \frac{\lambda}{|\lambda|}.$$

The limit $\lambda\to 0$ does not exist, since $\frac{\lambda}{|\lambda|}=1$ for $\lambda>0$ and $\frac{\lambda}{|\lambda|}=-1$ for $\lambda<0$.

Exercise 2.4. Suppose that $\Omega\subset\mathbb{R}^n$ is open, and $f,g:\Omega\to\mathbb{R}^m$ are differentiable at $p\in\Omega$. Show that $h=f+g$ is differentiable at p and

$$Dh(p) = Df(p)+Dg(p)$$

Solution: Since f and g are differentiable at p, there exist linear maps $Df(p), Dg(p):\mathbb{R}^n\to\mathbb{R}^m$ such that

$$\lim_{x\to p}\frac{\|f(x)-f(p)-Df(p)[x-p]\|}{\|x-p\|}=0,$$

and

$$\lim_{x\to p}\frac{\|g(x)-g(p)-Dg(p)[x-p]\|}{\|x-p\|}=0.$$

Now we estimate by the triangle inequality

$$\begin{aligned}\frac{\|h(x)-h(p)-Dh(p)[x-p]\|}{\|x-p\|} &= \frac{\|f(x)+g(x)-f(p)-g(p)-Df(p)[x-p]-Dg(p)[x-p]\|}{\|x-p\|}\\ &\le \frac{\|f(x)-f(p)-Df(p)[x-p]\|}{\|x-p\|}+\frac{\|g(x)-g(p)-Dg(p)[x-p]\|}{\|x-p\|},\end{aligned}$$

so that we have

$$\lim_{x\to p}\frac{\|h(x)-h(p)-Dh(p)[x-p]\|}{\|x-p\|}=0,$$

and the conclusion follows.

Exercise 2.5. Suppose $\Omega,\Omega'\subset\mathbb{R}^n$ are open, $g:\Omega\to\Omega'$ and $f:\Omega'\to\Omega$ are functions such that g is differentiable at $p\in\Omega$ and f is differentiable at $g(p)\in\Omega'$ and moreover

$$\begin{aligned} f\circ g(x) &= x, \qquad \forall\, x\in\Omega.\\ g\circ f(x) &= x, \qquad \forall\, x\in\Omega'.\end{aligned}$$

Show that

$$Df(g(p)) = (Dg(p))^{-1}.$$

Solution: We need to show that

$$Df(g(p)) \circ Dg(p) = \mathrm{id} \qquad \text{and} \qquad Dg(p) \circ Df(g(p)) = \mathrm{id}.$$

By the chain rule, we know that $f \circ g$ is differentiable at p. Moreover, since the identity is differentiable with derivative also the identity by a previous question, we deduce

$$\mathrm{id} = D(f \circ g)(p) = Df(g(p)) \circ Dg(p).$$

We also know by the chain rule that $g \circ f$ is differentiable at $g(p)$, with derivative

$$\mathrm{id} = D(g \circ f)(g(p)) = Dg(f(g(p))) \circ Df(g(p)) = Dg(p) \circ Df(g(p)),$$

where we have used $f(g(p)) = p$.

Exercise 2.6 (*)**.** (a) Show that the map $P : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$P : (x, y) \mapsto xy$$

is differentiable at each point $p = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{R}^2$, with Jacobian:

$$DP(p) = \begin{pmatrix} \eta & \xi \end{pmatrix}.$$

Solution: Let $h = (h_1, h_2)$

$$P(p+h) - P(p) = (\xi + h_1)(\eta + h_2) - \xi\eta = h_1\eta + h_2\xi + h_1 h_2,$$

so that:

$$P(p+h) - P(p) - DP(p)\,[h] = h_1 h_2.$$

Now, by Young's inequality we know that:

$$|h_1 h_2| \le \frac{1}{2}\left(h_1^2 + h_2^2\right) = \frac{1}{2}\|h\|^2,$$

so we have that:

$$\frac{|P(\xi + h_1, \eta + h_2) - P(\xi, \eta) - DP(p)\,[h]|}{\|h\|} \le \frac{1}{2}\|h\| \to 0,$$

as $\|h\| \to 0$.

(b) Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $q \in \mathbb{R}^n$. Show that the map $Q : \mathbb{R}^n \to \mathbb{R}^2$ given by:

$$Q : z \mapsto (f(z), g(z))$$

is differentiable at q and:

$$DQ(q) = \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix}$$

Solution: We calculate, for $h \in \mathbb{R}^n$:

$$Q(q+h) - Q(q) - DQ(q)[h] = \begin{pmatrix} f(q+h) \\ g(q+h) \end{pmatrix} - \begin{pmatrix} f(q) \\ g(q) \end{pmatrix} - \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix} h$$
$$= \begin{pmatrix} f(q+h) - f(q) - Df(q)[h] \\ g(q+h) - g(q) - Dg(q)[h] \end{pmatrix}.$$

Now, writing:

$$\begin{pmatrix} f(q+h) - f(q) - Df(q)[h] \\ g(q+h) - g(q) - Dg(q)[h] \end{pmatrix} = \begin{pmatrix} f(q+h) - f(q) - Df(q)[h] \\ 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ g(q+h) - g(q) - Dg(q)[h]t \end{pmatrix},$$

and applying the triangle inequality we have:

$$\|Q(q+h) - Q(q) - DQ(q)[h]\| \leq \|f(q+h) - f(q) - Df(q)[h]\|$$
$$+ \|g(q+h) - g(q) - Dg(q)[h]\|,$$

so that :

$$\lim_{h\to 0} \frac{\|Q(q+h) - Q(q) - DQ(q)[h]\|}{\|h\|} \leq \lim_{h\to 0} \frac{\|f(q+h) - f(q) - Df(q)[h]\|}{\|h\|}$$
$$+ \lim_{h\to 0} \frac{\|g(q+h) - g(q) - Dg(q)[h]\|}{\|h\|}$$
$$= 0.$$

(c) Show that $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(z) = f(z)g(z)$ for all $z \in \mathbb{R}^n$ is differentiable at q, and:

$$DF(q) = g(q)Df(q) + f(q)Dg(q)$$

[Hint: Note that $F = P \circ Q$.]

Solution: Note that $F = P \circ Q$. Since Q is differentiable at q and P is differentiable at $Q(q)$, by the chain rule we have that $P \circ Q$ is differentiable, at q, and moreover:

$$DF(q) = DP(Q(q)) \circ DQ(q) = \begin{pmatrix} g(q) & f(q) \end{pmatrix} \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix} = g(q)Df(q) + f(q)Dg(q).$$

Exercise 2.7. (a) Let the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x,y) = \begin{pmatrix} x^2 + e^{x+y} \\ x - \log y \\ 2xy + 1 \end{pmatrix}.$$

Assuming f is differentiable at a point $\begin{pmatrix} x \\ y \end{pmatrix}$, what is its derivative?

(b) Let $g : \mathbb{R}^3 \to \mathbb{R}^1$ be given by $g(x,y,z) = x + y + z$. Compute the derivative of $g \circ f$ assuming it exists. Compute it in 2 ways, with and without the chain rule.

Solution: (a) By the general formula for the Jacobian at $p = (x, y) \in \mathbb{R}^2$

$$Df(p) = \begin{pmatrix} D_1 f^1(p) & D_2 f^1(p) \\ D_1 f^2(p) & D_2 f^2(p) \\ D_1 f^3(p) & D_2 f^3(p) \end{pmatrix} = \left(\frac{\partial f^j}{\partial z_k} \right)_{j=1,2,3;k=1,2},$$

where we denoted $z_1 = x$, $z_2 = y$. Note that by convention in the matrix element notation a_{jk}, the first index refers to the row, and the second, to the column. Computing partial derivatives, we obtain

$$Df(p) = \begin{pmatrix} 2x + e^{x+y} & e^{x+y} \\ 1 & -\frac{1}{y} \\ 2y & 2x \end{pmatrix}.$$

(b) First, the derivative of g at a point q

$$Dg(q) = (D_1 f(q) \; D_2 f(q) \; D_3 f(q)) = (1 \; 1 \; 1).$$

Using Df from (a) and the chain rule, we obtain

$$D(g \circ f)(p) = (1 \; 1 \; 1) \times \begin{pmatrix} 2x + e^{x+y} & e^{x+y} \\ 1 & -\frac{1}{y} \\ 2y & 2x \end{pmatrix} = (2(x+y) + e^{x+y} + 1 \quad 2x + e^{x+y} - \frac{1}{y}).$$

Alternatively,

$$(g \circ f)(x, y, z) = x^2 + e^{x+y} + x - \log y + 2xy + 1,$$

and hence

$$D(g \circ f)(p) = (D_1(g \circ f)(p), D_2(g \circ f)(p)) = (2(x+y) + e^{x+y} + 1 \quad 2x + e^{x+y} - \frac{1}{y}).$$

Unseen Exercise. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} x^2 \sin(1/x) & \text{if } y = 0, x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Is the map f differentiable at $(0, 0) \in \mathbb{R}^2$? Justify your answer using the definition of the derivative.

Solution: Yes, the map f is differentiable at $(0, 0)$. We claim that $Df(0, 0)$ is the linear map $\Lambda \equiv 0$. To see this, let $h = (h^1, h^2) \in \mathbb{R}^2$. We note that

$$\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\| = \|f(h^1, h^2)\| \leq |h^1|^2.$$

Also, by an inequality in the exercises, we have $|h^1| \leq \|(h^1, h^2)\|$.

Thus,

$$\frac{\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\|}{\|(h^1, h^2)\|} \leq \frac{|h^1|^2}{|h^1|} = |h^1|.$$

This implies that

$$\lim_{(h^1, h^2) \to 0} \frac{\|f((0, 0) + (h^1, h^2)) - f(0, 0) - \Lambda[(h^1, h^2)]\|}{\|(h^1, h^2)\|} = 0.$$