**Exercise 2.1.** Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f(x) = x.$$

Show that $f$ is differentiable at each $p \in \mathbb{R}^n$ and

$$Df(p) = \mathrm{id},$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

**Exercise 2.2.** Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) \mapsto x^2 + y^2,$$

is differentiable at all points $p = (\xi, \eta) \in \mathbb{R}^2$ with Jacobian

$$Df(p) = (2\xi \;\; 2\eta)$$

**Exercise 2.3.** One might hope that the differential can be calculated by finding

$$\lim_{x \to p} \frac{f(x) - f(p)}{\|x - p\|}.$$

By considering the example of Exercise 2.1 or otherwise, show that this limit may not always exist, even if $f$ is differentiable at $p$.

**Exercise 2.4.** Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f, g : \Omega \to \mathbb{R}^m$ are differentiable at $p \in \Omega$. Show that $h = f + g$ is differentiable at $p$ and

$$Dh(p) = Df(p) + Dg(p)$$

**Exercise 2.5.** Suppose $\Omega, \Omega' \subset \mathbb{R}^n$ are open, $g : \Omega \to \Omega'$ and $f : \Omega' \to \Omega$ are functions such that $g$ is differentiable at $p \in \Omega$ and $f$ is differentiable at $g(p) \in \Omega'$ and moreover

$$\begin{aligned} f \circ g(x) &= x, \qquad \forall\, x \in \Omega. \\ g \circ f(x) &= x, \qquad \forall\, x \in \Omega'. \end{aligned}$$

Show that

$$Df(g(p)) = (Dg(p))^{-1}.$$

**Exercise 2.6 (*).** (a) Show that the map $P : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$P : (x, y) \mapsto xy$$

is differentiable at each point $p = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{R}^2$, with Jacobian:

$$DP(p) = (\eta \;\; \xi)\,.$$

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Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

(b) Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $q \in \mathbb{R}^n$. Show that the map $Q : \mathbb{R}^n \to \mathbb{R}^2$ given by:

$$Q : z \mapsto (f(z), g(z))$$

is differentiable at $q$ and:

$$DQ(q) = \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix}$$

(c) Show that $F : \mathbb{R}^n \to \mathbb{R}$ given by $F(z) = f(z)g(z)$ for all $z \in \mathbb{R}^n$ is differentiable at $q$, and:

$$DF(q) = g(q)Df(q) + f(q)Dg(q)$$

**Exercise 2.7.** (a) Let the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x, y) = \begin{pmatrix} x^2 + e^{x+y} \\ x - \log y \\ 2xy + 1 \end{pmatrix}.$$

Assuming $f$ is differentiable at a point $\begin{pmatrix} x \\ y \end{pmatrix}$, what is its derivative?

(b) Let $g : \mathbb{R}^3 \to \mathbb{R}^1$ be given by $g(x, y, z) = x + y + z$. Compute the derivative of $g \circ f$ assuming it exists. Compute it in 2 ways, with and without the chain rule.