Exercise 3.1. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is everywhere differentiable, and find the differential when:

(a) $f(x,y) = x^2 + y^2 - x - xy$,

(b) $f(x,y) = \frac{1}{\sqrt{1+x^2+y^2}}$,

(c) $f(x,y) = x^5y^2$.

Solution: (a) Computing the partial derivatives, we have (letting $p = (x,y)$):

$$D_1f(p) = 2x - 1 - y, \qquad D_2f(p) = 2y - x,$$

Clearly these are continuous at all $p \in \mathbb{R}^2$, so we deduce from the theorem in the lecture notes that f is everywhere differentiable and moreover:

$$Df(p) = (2x - 1 - y, 2y - x)$$

(b) Computing the partial derivatives, we have (letting $p = (x,y)$):

$$D_1f(p) = \frac{-x}{(1+x^2+y^2)^{\frac{3}{2}}}, \qquad D_2f(p) = \frac{-y}{(1+x^2+y^2)^{\frac{3}{2}}},$$

Clearly these are continuous at all $p \in \mathbb{R}^2$, so we deduce by the theorem in the lectures that that f is everywhere differentiable and moreover:

$$Df(p) = \frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}(-x, -y)$$

(c) Computing the partial derivatives, we have (letting $p = (x,y)$):

$$D_1f(p) = 5x^4y^2, \qquad D_2f(p) = 2x^5y,$$

Clearly these are continuous at all $p \in \mathbb{R}^2$, so we deduce from the theorem in the lectures that f is everywhere differentiable and moreover:

$$Df(p) = \left(5x^4y^2, 2x^5y\right)$$

Exercise 3.2. Suppose A is a symmetric $(n \times n)$ matrix. Consider the function:

$$\begin{array}{rccc} f & : & \mathbb{R}^n & \to & \mathbb{R} \\ & & x & \mapsto & xAx^t. \end{array}$$

(a) Show that f is differentiable at all points $p \in \mathbb{R}^n$, with:

$$Df(p) = 2pA$$

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Questions marked with $*$ are optional

(b) Find:

$$\operatorname{Hess} f(p).$$

Solution: (a) Fix $p \in \mathbb{R}^n$. We compute:

$$\begin{aligned} f(p+h) - f(p) - Df(p)[h] &= (p+h)A(p+h)^t - pAp^t - 2pAh^t \\ &= pAp^t + pAh^t + hAp^t + hAh^t - pAp^t - 2pAh^t \\ &= hAh^t, \end{aligned}$$

where we have used that A is symmetric to deduce $hAp^t = pAh^t$. Now, recall from an example in the lecture notes that for any matrix A there exists a constant K such that:

$$\left\|Ax^t\right\| \leq K \|x\|$$

for all $x \in \mathbb{R}^n$. Applying the Cauchy-Schwartz inequality we have:

$$\left\|hAh^t\right\| \leq \|h\| \left\|Ah^t\right\| \leq K \|h\|^2 .$$

Thus, we conclude:

$$\frac{\|f(p+h) - f(p) - Df(p)[h]\|}{\|h\|} \leq K \|h\| \to 0,$$

as $h \to 0$, thus we have that f is differentiable with derivative $Df(p) = 2pA$.

(b) If we write $A = (A_{ij})_{i,j=1}^n$, then we can write:

$$Df(p)[h] = \sum_{j=1}^n D_j f(p) h^j = 2 \sum_{i,j=1}^n p^i A_{ij} h^j$$

where $p = (p^1, \dots p^n)$, $h = (h^1, \dots, h^n)$. We deduce that:

$$D_j f(p) = 2 \sum_{i=1}^n p^i A_{ij}$$

Taking a further derivative, we conclude:

$$D_i D_j f(p) = 2A_{ij}.$$

Thus

$$\operatorname{Hess} f(p) = 2A.$$

Exercise 3.3. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f : (x, y, z) = xy^2 + x^2 + xze^y.$$

(a) Compute the first and second partial derivatives. Observe the properties of the second partial derivative.

(b) Write the terms of the Taylor expansion of f at zero up to and including the second-order terms.

(c) Without computation, write the same Taylor expansion up to and including the fourth-order terms.

Solution: (a) We have

$$D_1 f = y^2 + 2x + ze^y, \quad D_2 f = 2xy + xze^y, \quad D_3 f = xe^y.$$

Furthermore,

$$D_1D_1 f = 2, \quad D_2D_1 f = 2y + ze^y, \quad D_3D_1 f = e^y,$$

$$D_1D_2 f = 2y + ze^y, \quad D_2D_2 f = 2x + xze^y, \quad D_3D_2 f = xe^y,$$

$$D_1D_3 f = e^y, \quad D_2D_3 f = xe^y, \quad D_3D_3 f = 0.$$

(b) Thus by the general formula for the Taylor expansion, with $h_1 = x$, $h_2 = y$, $h_3 = z$,

$$\begin{aligned} f(x, y, z)) &= \sum_{\alpha, |\alpha| \le 2} D^\alpha f(0) \frac{h^\alpha}{\alpha!} + R_3 \\ &= f(0) + \sum_{j=1}^{3} D_j f(0) h_j + \sum_{j=1}^{3} D_j D_j f(0) \frac{h_j^2}{2!} + \sum_{j<k, j,k=1}^{3} D_j D_k f(0) h^j h^k + R_3 \\ &= x^2 + xz + R_3 \end{aligned}$$

(c)

$$f(x, y, z) = xy^2 + x^2 + xz(1 + y + y^2/2) + R_5$$

Exercise 3.4 (*)**.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{xy^3 - x^3y}{x^2 + y^2} & (x, y) \ne (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

(a) Show that:

$$D_1 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{y^3 - 3x^2y}{x^2 + y^2} - \dfrac{2x\left(xy^3 - x^3y\right)}{(x^2 + y^2)^2} & (x, y) \ne (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

and

$$D_2 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{3y^2x - x^3}{x^2 + y^2} - \dfrac{2y\left(xy^3 - x^3y\right)}{(x^2 + y^2)^2} & (x, y) \ne (0, 0) \\ 0 & (x, y) = (0, 0), \end{cases}$$

and show that these functions are both continuous at $(0, 0)$.

Solution: 1. Let $p = (x, y)$. If $p \neq 0$, we can differentiate using the quotient rule to find

$$D_1 f(p) = \frac{\partial f}{\partial x} = \frac{y^3 - 3x^2 y}{x^2 + y^2} - \frac{2x\left(xy^3 - x^3 y\right)}{\left(x^2 + y^2\right)^2}.$$

Further, note that $f(te_1) = 0$, so that:

$$\lim_{t \to 0} \frac{f(te_1) - f(0)}{t} = 0,$$

thus $D_1 f(0) = 0$.

2. Now, note that $\left|y^2 - 3x^2\right| \leq y^2 + 3x^2 \leq 3(y^2 + x^2)$, thus:

$$\left|\frac{y^3 - 3x^2 y}{x^2 + y^2}\right| = |y| \left|\frac{y^2 - 3x^2}{x^2 + y^2}\right| \leq 3\,|y|$$

Also, note that by Young's inequality $\left|xy^3\right| \leq \frac{1}{2}x^2y^2 + \frac{1}{2}y^4$ and similarly $\left|x^3 y\right| \leq \frac{1}{2}x^2y^2 + \frac{1}{2}x^4$, so that:

$$\left|xy^3 - x^3 y\right| \leq \left|xy^3\right| + \left|x^3 y\right| \leq \frac{1}{2}\left(x^4 + 2x^2y^2 + y^4\right) = \frac{1}{2}\left(x^2 + y^2\right)^2.$$

We deduce:

$$\left|\frac{2x\left(xy^3 - x^3 y\right)}{\left(x^2 + y^2\right)^2}\right| \leq |x|,$$

so that for $p = (x, y)^t \neq 0$, we have:

$$|D_1 f(p)| \leq 3\,|y| + |x| \to 0$$

as $p \to 0$, so that $D_1 f(p)$ is continuous at $p = 0$.

3. Similarly, if $p \neq 0$, we can differentiate using the quotient rule to find

$$D_2 f(p) = \frac{\partial f}{\partial y} = \frac{3y^2 x - x^3}{x^2 + y^2} - \frac{2y\left(xy^3 - x^3 y\right)}{\left(x^2 + y^2\right)^2}.$$

Further, note that $f(te_2) = 0$, so that:

$$\lim_{t \to 0} \frac{f(te_2) - f(0)}{t} = 0,$$

thus $D_2 f(0) = 0$.

4. Now, note that $\left|3y^2 - x^2\right| \leq 3y^2 + x^2 \leq 3(y^2 + x^2)$, thus:

$$\left|\frac{3y^2 x - x^3}{x^2 + y^2}\right| = |x| \left|\frac{3y^2 - x^2}{x^2 + y^2}\right| \leq 3\,|x|$$

Recalling that:

$$\left|xy^3 - x^3 y\right| \leq \frac{1}{2}\left(x^2 + y^2\right)^2.$$

We deduce:

$$\left|\frac{2y\left(xy^3 - x^3 y\right)}{\left(x^2 + y^2\right)^2}\right| \leq |y|,$$

so that for $p = (x, y) \neq 0$, we have:

$$|D_2 f(p)| \leq 3\,|y| + |x| \to 0$$

as $p \to 0$, so that $D_1 f(p)$ is continuous at $p = 0$.

(b) Show that:

$$\lim_{t\to 0} \frac{1}{t} \left(D_1 f(te_2) - D_1 f(0)\right) = 1$$

and

$$\lim_{t\to 0} \frac{1}{t} \left(D_2 f(te_1) - D_2 f(0)\right) = -1$$

Solution: We have (setting $x = 0$, $y = t$ in the formula for $D_1 f$):

$$D_1 f(te_2) = t, \quad D_1 f(0) = 0,$$

so that:

$$\lim_{t\to 0} \frac{1}{t} \left(D_1 f(te_2) - D_1 f(0)\right) = 1$$

Similarly, we have (setting $x = t$, $y = 0$ in the formula for $D_2 f$):

$$D_2 f(te_1) = -t, \quad D_1 f(0) = 0,$$

so that:

$$\lim_{t\to 0} \frac{1}{t} \left(D_2 f(te_1) - D_2 f(0)\right) = -1$$

(c) Conclude that both $D_2 D_1 f(0)$ and $D_1 D_2 f(0)$ exist, but that:

$$D_2 D_1 f(0) \neq D_1 D_2 f(0)$$

Solution: By definition,

$$D_2 D_1 f(0) = \lim_{t\to 0} \frac{1}{t} \left(D_1 f(te_2) - D_1 f(0)\right),$$

which certainly exists. Similarly,

$$D_1 D_2 f(0) = \lim_{t\to 0} \frac{1}{t} \left(D_2 f(te_1) - D_2 f(0)\right)$$

also exists, but as we've seen above the two are not equal.

Exercise 3.5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = e^x \sin(y)$.

a) Compute the degree 1 and degree 2 Taylor polynomial of f near the point $(x_0, y_0) = (0, \pi/2)$ and use those to approximate the value of f at $(x_1, y_1) = (0, \pi/2 + 1/4)$. Compare your results with the values you obtain from a calculator.

Solution: For all $x, y \in \mathbb{R}$ we have

$$\begin{aligned} &D_1 f(x, y) = e^x \sin(y), && D_1 D_1 f(x, y) = e^x \sin(y), && D_2 D_1 f(x, y) = e^x \cos(y) \\ &D_2 f(x, y) = e^x \cos(y), && D_2 D_2 f(x, y) = -e^x \sin(y), && D_1 D_2 f(x, y) = e^x \cos(y). \end{aligned}$$

Evaluating the above expressions at $(x_0, y_0) = (0, \pi/2) \in \mathbb{R}^2$, we get $f(x_0, y_0) = 1$ as well as

$$\begin{aligned} &D_1 f(0, \pi/2) = 1, && D_1 D_1 f(0, \pi/2) = 1, && D_2 D_1 f(0, \pi/2) = 0 \\ &D_2 f(0, \pi/2) = 0, && D_2 D_2 f(0, \pi/2) = -1, && D_1 D_2 f(0, \pi/2) = 0. \end{aligned}$$

The Taylor polynomials $T_1 f$ and $T_2 f$ of degree 1 and 2, respectively, are therefore

$$T_1 f(x,y) = f(x_0,y_0) + D_1 f(x_0,y_0)\cdot(x-x_0) + D_2 f(x_0,y_0)\cdot(y-y_0) = 1+x$$

and

$$\begin{aligned}T_2 f(x,y) &= T_1 f(x,y) + \frac{1}{2}\Big[D_1D_1 f(x_0,y_0)\cdot(x-x_0)^2 + D_2D_2 f(x_0,y_0)\cdot(y-y_0)^2 \\ &\quad + 2D_1D_2 f(x_0,y_0)\cdot(x-x_0)(y-y_0)\Big] \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}(y-\pi/2)^2.\end{aligned}$$

At the point $(x_1,y_1) = (0,\pi/2+1/4)$, these yield the approximations

$$T_1 f(0,\pi/2+1/4) = 1, \quad T_2 f(0,\pi/2+1/4) = 1 - 1/2(1/4)^2 = 31/32 = 0.96875.$$

The approximation by T_2 is very good as the actual value (using a high precision calculator) is

$$f(0,\pi/2+1/4) \approx 0.96891.$$

b) How precise is the degree 1 approximation in the closed ball of radius 1/4 around (x_0,y_0). Find a rigorous upper bound for the approximation error.

Solution: Let B denote the ball of radius 1/4 about (x_0,y_0), that is $B_{1/4}(x_0,y_0)$. By Theorem 1.14, the remainder term $R_1 = f - T_1 f$ can be expressed as

$$\begin{aligned}R_1(x,y) = \frac{1}{2}\Big[&D_1D_1 f(x_r,y_r)\cdot(x-x_0)^2 \\ &+ D_2D_2 f(x_r,y_r)\cdot(y-y_0)^2 + 2D_1D_2 f(x_r,y_r)\cdot(x-x_0)(y-y_0)\Big]\end{aligned}$$

for some (x_r,y_r) such that x_r lies in the interval $[x_0,x]$ when $x > x_0$ and in the interval $[x,x_0]$ when $x \le x_0$, and similarly for y_r. In particular, for all $(x,y) \in B_{1/4}(x_0,y_0)$, this gives $|x_r - x_0| \le 1/4$ and $|y_r - y_0| \le 1/4$. Moreover, by part a) for all $(x,y) \in \mathbb{R}^2$ we have $|D_1D_1 f(x,y)| \le e^x$, $|D_1D_2 f(x,y)| \le e^x$, and $|D_2D_2 f(x,y)| \le e^x$, using $|\sin(x)| \le 1$ and $|\cos(x)| \le 1$. Overall, this gives for all $(x,y) \in B_{1/4}(x_0,y_0)$,

$$|R_1(x,y)| \le 4\big(\tfrac{1}{2}e^{\frac{1}{4}}\cdot(\tfrac{1}{4})^2\big) = \tfrac{1}{8}e^{\frac{1}{4}} \approx 0.1605.$$

Even the (relatively crude) first-order approximation is off by at most about 16% of the value at (x_0,y_0) in $B_{1/4}(x_0,y_0)$.

Unseen Exercise. Find the minimum of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f(x,y,z) = x^4(y^2+x^2) + z^2 - 4z$$

Solution: Computing the partial derivatives, we have (setting $p = (x,y,z)$)

$$\begin{aligned}D_1 f(p) &= 4x^3y^2 + 6x^5 = x^3(4y^2+6x^2) \\ D_2 f(p) &= 2yx^4 \\ D_3 f(p) &= 2z - 4\end{aligned}$$

We see that all partial derivatives are continuous, thus f is everywhere differentiable. If $p_0 = (x_0, y_0, z_0)$ is an extremal point, then $Df(p_0) = 0$. This implies that either

$$(x_0, y_0, z_0) = (0, 0, 2),$$

or

$$(x_0, y_0, z_0) = (0, y, 2),$$

for any value of $y \in \mathbb{R}$. In either of the above cases $f(p_0) = -4$. To see this is a minimum, note that

$$f(p) = x^4(y^2 + x^2) + z^2 - 4z = x^4(y^2 + x^2) + (z - 2)^2 - 4 \geq -4,$$

since the first two terms are manifestly positive.