Exercise 3.1. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is everywhere differentiable, and find the differential when:

(a) $f(x, y) = x^2 + y^2 - x - xy$,

(b) $f(x, y) = \frac{1}{\sqrt{1+x^2+y^2}}$,

(c) $f(x, y) = x^5 y^2$.

Exercise 3.2. Suppose A is a symmetric $(n \times n)$ matrix. Consider the function:

$$\begin{array}{rcccl} f & : & \mathbb{R}^n & \to & \mathbb{R} \\ & & x & \mapsto & xAx^t. \end{array}$$

(a) Show that f is differentiable at all points $p \in \mathbb{R}^n$, with:

$$Df(p) = 2pA$$

(b) Find:

$$\text{Hess}\, f(p).$$

Exercise 3.3. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f : (x, y, z) = xy^2 + x^2 + xze^y.$$

(a) Compute the first and second partial derivatives. Observe the properties of the second partial derivative.

(b) Write the terms of the Taylor expansion of f at zero up to and including the second-order terms.

(c) Without computation, write the same Taylor expansion up to and including the fourth-order terms.

Exercise 3.4 (*)**.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{xy^3 - x^3y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

(a) Show that:

$$D_1 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{y^3 - 3x^2y}{x^2 + y^2} - \dfrac{2x\left(xy^3 - x^3y\right)}{\left(x^2 + y^2\right)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

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Questions marked with $*$ are optional

and

$$D_2 f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{3y^2x - x^3}{x^2+y^2} - \dfrac{2y\left(xy^3 - x^3y\right)}{\left(x^2+y^2\right)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0), \end{cases}$$

and show that these functions are both continuous at $(0,0)$.

(b) Show that:

$$\lim_{t\to 0} \frac{1}{t}\left(D_1 f(te_2) - D_1 f(0)\right) = 1$$

and

$$\lim_{t\to 0} \frac{1}{t}\left(D_2 f(te_1) - D_2 f(0)\right) = -1$$

(c) Conclude that both $D_2D_1f(0)$ and $D_1D_2f(0)$ exist, but that:

$$D_2D_1f(0) \neq D_1D_2f(0)$$

Exercise 3.5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = e^x \sin(y)$.

a) Compute the degree 1 and degree 2 Taylor polynomial of f near the point $(x_0, y_0) = (0, \pi/2)$ and use those to approximate the value of f at $(x_1, y_1) = (0, \pi/2 + 1/4)$. Compare your results with the values you obtain from a calculator.

b) How precise is the degree 1 approximation in the closed ball of radius $1/4$ around (x_0, y_0). Find a rigorous upper bound for the approximation error.