Exercise 4.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y - xy \\ x^2 \end{pmatrix}$$

Determine the set of points in $\mathbb{R}^2$ such that f is invertible near those points, and compute the derivative of the inverse map.

Solution: The derivative is

$$Df = \begin{pmatrix} 1-y & 1-x \\ 2x & 0 \end{pmatrix}.$$

We have $\det Df = 2x(x-1)$ which is zero if $x = 0$ or $x = 1$ for any y. Thus, for any $(x, y) \in \mathbb{R}^2$ such that $x \notin \{0, 1\}$, the function is invertible on a ball around $(x, y) \in \mathbb{R}^2$, and the derivative of the inverse is

$$Df^{-1} = (Df)^{-1} = \frac{1}{2x(x-1)} \begin{pmatrix} 0 & x-1 \\ -2x & 1-y \end{pmatrix}.$$

Exercise 4.2. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighbourhood of the origin, and $f'(0) = 0$. Give an example to show that f may nevertheless be bijective.

[Hint: Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto x^3$.]

Solution: The function $f : x \mapsto x^3$ is strictly monotone increasing and continuous, hence it is bijective. On the other hand $f'(0) = 0$.

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, differentiable at the origin, and $\det Df(0) = 0$. Show that f^{-1} is not differentiable at $f(0)$.

[Hint: Assume that f^{-1} is differentiable at $f(0)$ and apply the chain rule to $\iota = f^{-1} \circ f = f \circ f^{-1}$ to derive a contradiction.]

Solution: Assume that f^{-1} is differentiable at $f(0)$ and let us apply the chain rule to differentiate $\iota = f^{-1} \circ f$ at 0. We find

$$\iota = Df^{-1}(f(0)) \circ Df(0).$$

Similarly, applying the chain rule to differentiate $\iota = f \circ f^{-1}$ at $f(0)$, we have:

$$\iota = Df(f^{-1}(f(0))) \circ Df^{-1}(f(0)) = Df(0) \circ Df^{-1}(f(0)).$$

We conclude that $Df(0)$ has both a left and right inverse and thus is invertible, however $\det Df(0) = 0$. This contradicts the assumption that f^{-1} is differentiable at $f(0)$.

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

Exercise 4.3. The non-linear system of equations

$$\begin{aligned} e^{xy}\sin(x^2 - y^2 + x) &= 0 \\ e^{x^2+y}\cos(x^2 + y^2) &= 1 \end{aligned}$$

admits the solution $(x, y) = (0, 0)$. Prove that there exists $\varepsilon > 0$ such that for all (ξ, η) with $\xi^2 + \eta^2 < \varepsilon^2$, the perturbed system of equations

$$\begin{aligned} e^{xy}\sin(x^2 - y^2 + x) &= \xi \\ e^{x^2+y}\cos(x^2 + y^2) &= 1 + \eta \end{aligned}$$

has a solution $(x(\xi, \eta), y(\xi, \eta))$ which depends continuously on (ξ, η).

Solution: Let us define the maps

$$f^1(x, y) = e^{xy}\sin(x^2 - y^2 + x), \qquad f^2(x, y) = e^{x^2+y}\cos(x^2 + y^2),$$

for $(x, y) \in \mathbb{R}^2$. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x, y) = \begin{pmatrix} f^1(x, y) \\ f^2(x, y) \end{pmatrix} = \begin{pmatrix} e^{xy}\sin(x^2 - y^2 + x) \\ e^{x^2+y}\cos(x^2 + y^2) \end{pmatrix}.$$

Then we have $f(0, 0) = (0, 1)$. We aim to employ the Inverse Function Theorem.

We compute the first partial derivatives of F, as

$$\begin{aligned} D_1 f^1(x, y) &= ye^{xy}\sin(x^2 - y^2 + x) + (2x + 1)e^{xy}\cos(x^2 - y^2 + x) \\ D_2 f^2(x, y) &= xe^{xy}\sin(x^2 - y^2 + x) + 2ye^{xy}\cos(x^2 - y^2 + x) \\ D_1 f^2(x, y) &= 2xe^{x^2+y}\cos(x^2 + y^2) - 2xe^{x^2+y}\sin(x^2 + y^2) \\ D_2 f^2(x, y) &= e^{x^2+y}\cos(x^2 + y^2) - 2ye^{x^2+y}\cos(x^2 + y^2) \end{aligned}$$

All these partial derivatives are continuous, so by a theorem in the lectures, f is continuously differentiable. Moreover, we have

$$Df(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is invertible. Thus, by the Inverse Function Theorem, there exists a neighbourhoods $U \subset \mathbb{R}^2$ of $(0, 0)$ and a neighbourhood $V \subset \mathbb{R}^2$ of $(0, 1)$ such that $f : U \to V$ is a bijection.

Since V is an open neighbourhood of $(0, 1)$, there is $\epsilon > 0$ such that $B_\epsilon(0, 1) \subseteq V$. It follows that all the points $(\xi, 1 + \eta)$ with $\xi^2 + \eta^2 < \varepsilon^2$ are elements of V. Thus, the inverse map

$$(x(\xi, \eta), y(\xi, \eta)) = f^{-1}(\xi, 1 + \eta)$$

is well-defined and solves the perturbed system. The continuity of the map f^{-1} implies that $x(\xi, \eta)$ and $y(\xi, \eta)$ each vary continuously in (ξ, η) (see Exercise 1.8(b) on Problem Sheet 1).

Exercise 4.4. For each of the following equations determine at which points one cannot find a function $y = f(x)$ which describes the graph in this neighbourhood. Sketch the graphs.

(a)

$$\frac{1}{3}y^3 - 2y + x = 1$$

(b)

$$x^2\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}\right) - xy\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\sin(2\phi) + y^2\left(\frac{\sin^2\phi}{a^2}+\frac{\cos^2\phi}{b^2}\right) = 1,$$

where $a > 0$, $b > 0$, $0 \le \phi \le \pi/2$ are fixed parameters. Note the cases $a = b$, $\phi = 0$, $\phi = \pi/2$.

Solution: (a) Let

$$F(x,y) = \frac{1}{3}y^3 - 2y + x - 1.$$

The solutions of the equation satisfy $F(x,y) = 0$. To employ the Implicit Function Theorem, we need to identify the solutions (x,y) of $F(x,y) = 0$ such that $\frac{\partial}{\partial y}F(x,y) \neq 0$. Solving the equation $\frac{\partial}{\partial y}F(x,y) = 0$ gives $y = \pm\sqrt{2}$. Substituting $y = +\sqrt{2}$ in $F(x,y) = 0$ we get $x = 1 - \frac{4}{3}\sqrt{2}$, and substituting $y = -\sqrt{2}$ in $F(x,y) = 0$ we get $x = 1 + \frac{4}{3}\sqrt{2}$. Thus, the theorem does not apply at the points

$$(1 - \frac{4}{3}\sqrt{2}, \sqrt{2}), \qquad (1 + \frac{4}{3}\sqrt{2}, -\sqrt{2}).$$

Now by the Implicit Function Theorem, for every (x,y) in $\mathbb{R}^2$, except the above two points, the solution of the equation $F(x,y) = 0$ near (x,y) is the graph of a function. That is, given (x,y), there are open sets A containing x and an open set B containing y, and a function $g : A \to B$ such that $(x', y') \in A \times B$ is a solution of the equation $F(x', y') = 0$ if and only if $y' = g(x')$.

To see what is happening at the two exceptional points, we may rewriting the equation in the form

$$x = -\frac{1}{3}y^3 + 2y + 1.$$

We note that the first derivative $\frac{d}{dy}x = 0$ and the second derivative $\frac{d^2}{dy^2}x \neq 0$ at any of the two exceptional points. Thus, those points are either a maximum or minimum for the graph of the function which gives the solution in terms of y. Thus, $y = g(x)$ does not exist in any neighbourhood.

(b) As in the previous part, we may write the equation in the form $F(x,y) = 0$, for a suitable function F. The candidate points where the Implicit Function Theorem cannot be applied are the solutions of the equation $\frac{\partial}{\partial y}F(x,y) = 0$. That gives us

$$y = x\frac{b^2 - a^2}{b^2\sin^2\phi + a^2\cos^2\phi}\frac{\sin(2\phi)}{2}.$$

If we substitute the above relation in the equation $F(x,y) = 0$, we obtain 2 points on the graph (one for the pluses signs and one for the minuses signs):

$$x = \pm\sqrt{b^2\sin^2\phi + a^2\cos^2\phi}, \qquad y = \pm\frac{b^2 - a^2}{\sqrt{b^2\sin^2\phi + a^2\cos^2\phi}}\frac{\sin(2\phi)}{2}.$$

Note that the solution of the equation $F(x,y)=0$ is in fact the ellipse

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$$

rotated by the angle ϕ, using the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus, at the two points we have identified, the solution cannot be written as the graph of a function. Indeed, for $a=b$, this problem reduces to the one we considered in the lectures.

Exercise 4.5. Consider the equation

$$2x^2+4xy+y^2=3x+4y$$

a) Show that this system of equations (implicitly) defines a function $y=f(x)$ with $f(1)=1$.

Solution: We consider the function $F:\mathbb{R}^2\to\mathbb{R}$ defined as

$$F(x,y)=(2x^2+4xy+y^2)-(3x+4y).$$

We note that $F(1,1)=0$, that is, $(x_0,y_0)=(1,1)$ is a solution of the equation $F(x,y)=0$. We aim to employ the Implicit Function Theorem.

We have

$$D_2F(x,y)=4x+2y-4,$$

which shows that D_2F is a continuous function. Moreover, $D_2F(1,1)=2\neq 0$.

By the (simple version of the) Implicit Function Theorem, there exists a neighbourhood $U\subset\mathbb{R}$ of $x_0=1$ and a continuously differentiable function $f:U\to\mathbb{R}$ satisfying $f(1)=f(x_0)=y_0=1$ such that

$$F(x,f(x))=0 \text{ for all } x\in U.$$

b) Compute $f'(1)$ without knowing f explicitly.

Solution: Let us consider the map $g(x)=F(x,f(x))$, for $x\in U$. We may write this map as the composition of the maps $h(x)=(x,f(x))$ followed by the map $F(x,y)$. That is, $g(x)=F\circ h(x)$. By the chain rule, we have

$$\begin{aligned} Dg(x)=DF(h(x))\circ Dh(x) &= \begin{pmatrix} D_1F(x,f(x)) & D_2F(x,f(x)) \end{pmatrix} \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} \\ &= D_1F(x,f(x))+D_2F(x,f(x))f'(x). \end{aligned}$$

From the definition of the function F, we have

$$D_1F(x,y)=4x+4y-3,$$

and hence $D_1F(1,1)=5$. On the other hand, since $g\equiv 0$ on U, we have $g'(1)=0$. Therefore, the above equation at $x=1$ gives us

$$0=5+2f'(1),$$

which implies $f'(1)=-5/2$.

c) Find an explicit formula for f and check your result from b).

Solution: To identify f explicitly, we must solve the equation $F(x, y) = 0$ for y, which is possible here since F is a quadratic equation. That gives us

$$y = 2 - 2x \pm \sqrt{2x^2 - 5x + 4}.$$

Since $f(1) = 1 > 0$ we must choose the positive sign in the above equation, which becomes

$$f(x) = 2 - 2x + \sqrt{2x^2 - 5x + 4}.$$

It follows that

$$f'(x) = -2 + \frac{4x - 5}{2\sqrt{2x^2 - 5x + 4}},$$

and hence $f'(1) = -2 - 1/2 = -5/2$.

Unseen Exercise. (unseen) Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Consider the function $f : \Omega \to \mathbb{R}^2$ given by:

$$f : (x, y) = (x \sin y, x \cos y).$$

(a) Show that f is differentiable at all $p = (\xi, \eta) \in \Omega$, with:

$$Df(p) = \begin{pmatrix} \sin\eta & \xi\cos\eta \\ \cos\eta & -\xi\sin\eta \end{pmatrix}.$$

Solution: Let $f^1(x, y) = x \sin y$ and $f^2(x, y) = x \cos y$. We can compute the partial derivatives at p and find

$$D_1 f^1(p) = \sin\eta, \qquad D_2 f^1(p) = \xi\cos\eta,$$
$$D_1 f^2(p) = \cos\eta, \qquad D_2 f^2(p) = -\xi\sin\eta.$$

These are all manifestly continuous functions of p, so we deduce that f is everywhere differentiable and:

$$Df(p) = \begin{pmatrix} \sin\eta & \xi\cos\eta \\ \cos\eta & -\xi\sin\eta \end{pmatrix},$$

by the theorem in the lectures.

(b) Show that $Df(p)$ is invertible for all $p \in \Omega$.

Solution: We have $\det Df(p) = -\xi \neq 0$ for $p = (\xi, \eta) \in \Omega$. Thus $Df(p)$ is invertible for all $p \in \Omega$.

(c) Show that $f : \Omega \to \mathbb{R}^2$ is not injective. Deduce that the restriction to open sets U, V in the inverse function theorem is necessary.

Solution: f is not injective, since (for example) the points $(1, 0)$ and $(1, 2\pi)$ are both mapped to $(0, 1)$ under f. This shows that even for a function whose derivative is globally invertible, we can nevertheless have that the function is not globally injective. Locally (i.e. restricted to small enough open sets) we do recover injectivity.