Exercise 4.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y - xy \\ x^2 \end{pmatrix}$$

Determine the set of points in $\mathbb{R}^2$ such that f is invertible near those points, and compute the derivative of the inverse map.

Exercise 4.2. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighbourhood of the origin, and $f'(0) = 0$. Give an example to show that f may nevertheless be bijective.

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, differentiable at the origin, and $\det Df(0) = 0$. Show that f^{-1} is not differentiable at $f(0)$.

Exercise 4.3. The non-linear system of equations

$$e^{xy} \sin(x^2 - y^2 + x) = 0$$
$$e^{x^2+y} \cos(x^2 + y^2) = 1$$

admits the solution $(x, y) = (0, 0)$. Prove that there exists $\varepsilon > 0$ such that for all (ξ, η) with $\xi^2 + \eta^2 < \varepsilon^2$, the perturbed system of equations

$$e^{xy} \sin(x^2 - y^2 + x) = \xi$$
$$e^{x^2+y} \cos(x^2 + y^2) = 1 + \eta$$

has a solution $(x(\xi, \eta), y(\xi, \eta))$ which depends continuously on (ξ, η).

Exercise 4.4. For each of the following equations determine at which points one cannot find a function $y = f(x)$ which describes the graph in this neighbourhood. Sketch the graphs.

(a)

$$\frac{1}{3} y^3 - 2y + x = 1$$

(b)

$$x^2 \left(\frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2} \right) - xy \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \sin(2\phi) + y^2 \left(\frac{\sin^2 \phi}{a^2} + \frac{\cos^2 \phi}{b^2} \right) = 1,$$

where $a > 0$, $b > 0$, $0 \le \phi \le \pi/2$ are fixed parameters. Note the cases $a = b$, $\phi = 0$, $\phi = \pi/2$.

Exercise 4.5. Consider the equation

$$2x^2 + 4xy + y^2 = 3x + 4y$$

a) Show that this system of equations (implicitly) defines a function $y = f(x)$ with $f(1) = 1$.

b) Compute $f'(1)$ without knowing f explicitly.

c) Find an explicit formula for f and check your result from b).

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional