Exercise 5.1. Let $X = \mathbb{R}^n$ and define the function $\mathrm{d}_{infty} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\mathrm{d}_\infty(x, y) = \max\{|x^1 - y^1|, \dots, |x^n - y^n|\}.$$

Show that d_∞ is a metric on $\mathbb{R}^n$.

Exercise 5.2. Show that each of the following functions is a metric on $\mathbb{R}$:

(i) $\mathrm{d}(x, y) = |x^3 - y^3|$, (here x^3 means x raised to power 3)

(ii) $\mathrm{d}(x, y) = |e^x - e^y|$,

(iii) $\mathrm{d}(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.

Which property of the maps $x \mapsto x^3$, $x \mapsto e^x$, and $x \mapsto \tan^{-1}(x)$ makes these functions a metric.

Exercise 5.3. Assume that $a < b$ are real numbers, and $h : (a, b) \to (0, \infty)$ is a continuous function. For x and y in (a, b), we define

$$\mathrm{d}_h(x, y) = \int_{\min\{x,y\}}^{\max\{x,y\}} h(t)\, dt.$$

Show that d_h is a metric on (a, b).

Exercise 5.4. Consider the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$g(x, y) = |x - y|^2.$$

Show that g is not a metric on $\mathbb{R}$.

Exercise 5.5. Let $X = \mathbb{R}^2$, and define $\mathrm{d}_{\mathrm{rail}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$\mathrm{d}_{\mathrm{rail}}(x, y) = \begin{cases} \|x - y\| & \text{if } x = ky \text{ for some } k \in \mathbb{R} \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

Show that $\mathrm{d}_{\mathrm{rail}}$ is a metric on $\mathbb{R}^2$.

This is called the British rail metric. The intuition behind this metric is that if two towns are on the same rail line, then we travel between them, but if the towns are on distinct lines, we travel via London (represented as the origin in $\mathbb{R}^2$).

Exercise 5.6. Assume that $a < b$ are real numbers. Show that each of the following functions is a norm on $C([a, b])$:

(i)

$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

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Questions marked with $*$ are optional

(ii)

$$\|f\|_\infty = \max_{t\in[a,b]} |f(t)|$$

(iii)

$$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt\right)^{1/2}$$

Exercise 5.7. Show that if V is a vector space, and $\|\cdot\| : V \to \mathbb{R}$ is a norm function, then for any $v \in V$, we must have $\mathrm{d}_{\|\|}(0, 2v) = 2\,\mathrm{d}_{\|\|}(0, v)$. Conclude that there is no norm function on $\mathbb{R}^2$ which induced the discrete metric d_{disc} on $\mathbb{R}^2$.

Exercise 5.8. Let (X, d) be a metric space.

(i) Show that for every x, y, and z in X, we have

$$|\,\mathrm{d}(x,z) - \mathrm{d}(y,z)| \le \mathrm{d}(x,y).$$

(ii) Show that for all x, y, z and t in X, we have

$$|\,\mathrm{d}(x,y) - \mathrm{d}(z,t)| \le \mathrm{d}(x,z) + \mathrm{d}(y,t).$$

(iii) Show that for all x_1, x_2, $\dots$, x_n in X, we have

$$\mathrm{d}(x_1,x_n) \le \mathrm{d}(x_1,x_2) + \mathrm{d}(x_2,x_3) + \cdots + \mathrm{d}(x_{n-1},x_n).$$

Exercise 5.9. Let (X, d) be a metric space.

(i) Show that if $\epsilon < \delta$, then $B_\epsilon(x) \subseteq B_\delta(x)$. By an example, show that the equality may hold even if $\epsilon < \delta$.

(ii) Show that for every $x \in X$, we have

$$\bigcap_{n\in\mathbb{N}} B_{1/n}(x) = \{x\}.$$

Exercise 5.10. (i) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x,y) \le \mathrm{d}_2(x,y) \le \sqrt{n}\cdot \mathrm{d}_\infty(x,y).$$

(ii) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x,y) \le \mathrm{d}_1(x,y) \le n\cdot \mathrm{d}_\infty(x,y).$$

(iii) Show/conclude that for all x and y in $\mathbb{R}^n$, we have

$$\frac{1}{\sqrt{n}}\,\mathrm{d}_2(x,y) \le \mathrm{d}_1(x,y) \le \sqrt{n}\,\mathrm{d}_2(x,y).$$

(iv) Conclude that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ are topologically equivalent.