Exercise 6.1. Let $(X, \mathrm{d}_{\text{disc}})$ be a discrete metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Then, $(x_n)_{n\geq 1}$ converges in $(X, \mathrm{d}_{\text{disc}})$ if and only if the sequence $(x_n)_{n\geq 1}$ is eventually constant.

Solution: Assume $(x_n)_{n\geq 1}$ converges to $x \in (X, \mathrm{d}_{\text{disc}})$. Then $\forall \epsilon > 0$ there is N s.t. $\forall n > N$, $x_n \in B_\epsilon(x)$. Take for example $\epsilon = 1/2$. Since in our space $B_{1/2}(x) = \{x\}$, we have $x_n = x$, $\forall n > N$. In other words, the sequence is eventually constant.

For the opposite implication, assume that there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n = x_N$. Then, for all $\epsilon > 0$, $x_n \in B_\epsilon(x_n)$. Thus, for all $\epsilon > 0$, and all $n \geq N$, $x_n \in B_\epsilon(x_N)$. This implies that the sequence $(x_n)_{n\geq 1}$ converges to x_N.

Exercise 6.2. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Prove that the sequence $(x_n)_{n\geq 1}$ converges to $x \in X$ if and only if, for every open set U in (X, d) with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.

Hint: U can be the ball $B_r(x)$.

Solution: Assume that $(x_n)_{n\geq 1}$ converges to $x \in X$. Let U be an arbitrary open set which contains x. Since U is open and $x \in U$, there is $\delta > 0$ such that $B_\delta(x) \subset U$. Since $(x_n)_{n\geq 1}$ converges to x, for δ there is $N = N(\delta)$ such that for all $n \geq N$ we have $x_n \in B_\delta(x)$. Since $B_\delta(x) \subset U$, for all $n \geq N$ we have $x_n \in U$.

For the opposite implication assume that $(x_n)_{n\geq 1}$ is a sequence in X and for any open set $U \subset X$ with $x \in U$, there is N such that for all $n \geq N$, we have $x_n \in U$. Fix an arbitrary $\epsilon > 0$ and define $U = B_\epsilon(x)$ (recall that any ball is an open set). By the hypothesis, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in U = B_\epsilon(x)$. As ϵ was arbitrary, we conclude that $(x_n)_{n\geq 1}$ converges to x.

Exercise 6.3. Let $(X, \mathrm{d}_{\text{disc}})$ be a discrete metric space. Then every set in X is closed.

Hint: First show that every set in X is open with respect to d_{disc}.

Solution: We show first that every set in X is open. Let us fix an arbitrary set $A \subset X$. For any $x \in A$, we have $x \in B_{1/2}(x) = \{x\} \subset A$. By definition, this means that A is open. Thus any set in X is open. Now take an arbitrary set $B \in X$. We have just shown that $X \setminus B$ is open. Therefore, by a theorem in the lectures, B is closed.

Exercise 6.4. Let (X, d) be a metric space, and V be a subset of X. Show that the set V is closed if and only if $\overline{V} = V$.

Hint: use the definition of closed sets, and the definition of the closure of a set.

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Questions marked with $*$ are optional

Solution: By definition, $\overline{V}$ is the union of V with all its accumulation points.

First assume that V is closed. Then, by definition, for any sequence $(x_n)_{n\geq 1}$ in V which converges to $x \in X$, x belongs to V. We need to show that $\overline{V} = V$, for which it suffices to verify that all accumulation points of V belong to V. Let a be an accumulation point of V. Then, for any $\epsilon > 0$ the set $B_\epsilon(a) \cap V$ contains a point of V different from a. For a natural number n, let $x_n \in B_{1/n}(a) \cap V$. Then, x_n, $n = 1, 2, \dots$, is a sequence of points in V converging to a. Since we assumed V closed, we must have $a \in V$. As a was an arbitrary accumulation point of V, we conclude that $\overline{V} \subset V$.

On the other hand, assume that $\overline{V} = V$. Assume that $(x_n)_{n\geq 1}$ be an arbitrary sequence in V which converges to some $x \in X$. To show that V is closed, we need to verify that $x \in V$. If there is $n_0 \in \mathbb{N}$ such that $x = x_{n_0}$, then we are done. So let us assume that for all $n \in \mathbb{N}$, we have $x \neq x_n$. Since $(x_n)_{n\geq 1}$ belongs to V and converge to $x \in X$, any ball centred at x contains some element x_n, which we assumed different from x. Thus, x is an accumulation point of V, and since $\overline{V} = V$, $x \in V$.

Unseen Exercise. (unseen) Let V and W be subsets of a metric space (X, d). The following properties hold:

(i) if $V \subset W$, then $V^\circ \subset W^\circ$,

(ii) if $V \subset W$, then $\overline{V} \subset \overline{W}$,

Solution: Let $V \subset W$.

(i) Fix an arbitrary $x \in V^\circ$. There is $\epsilon > 0$ such that $B_\epsilon(x) \subset V$. As $V \subset W$, we have $B_\epsilon(x) \subset W$. Thus, there is $\epsilon > 0$ such that $B_\epsilon(x) \subset W$. This means that $x \in W^\circ$.

(ii) Fix an arbitrary $x \in \overline{V}$. Then, for every $\delta > 0$, $B_\delta(x) \cap V \neq \emptyset$. As $V \subset W$, for every $\delta > 0$, $B_\delta(x) \cap W \neq \emptyset$. This means that $x \in \overline{W}$.

Exercise 6.5. Let V and W be subsets of a metric space (X, d). Prove that

$$\overline{V \cup W} = \overline{V} \cup \overline{W}.$$

Give an example of (X, d), V and W such that

$$(V \cup W)^\circ \neq V^\circ \cup W^\circ.$$

Solution: We first show $\overline{V \cup W} \subset \overline{V} \cup \overline{W}$. Let us fix an arbitrary $x \in \overline{V \cup W}$. Suppose $x \notin \overline{V} \cup \overline{W}$. Then, $x \notin \overline{V}$ and $x \notin \overline{W}$. As $x \notin \overline{V}$, there is $\epsilon_1 > 0$ such that $B_{\epsilon_1}(x) \cap V = \emptyset$, and since $x \notin \overline{W}$, there is $\epsilon_2 > 0$ such that $B_{\epsilon_2}(x) \cap W = \emptyset$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then, $B_\epsilon(x) \cap (V \cup W) = \emptyset$. This contradicts $x \in \overline{V \cup W}$.

Now we show that $\overline{V} \cup \overline{W} \subset \overline{V \cup W}$. Fix an arbitrary $x \in \overline{V} \cup \overline{W}$. Then either $x \in \overline{V}$, or $x \in \overline{W}$. If $x \in \overline{V}$, then for every $\delta > 0$, $B_\delta(x) \cap V \neq \emptyset$. This implies that for every $\delta > 0$, $B_\delta(x) \cap (V \cup W) \neq \emptyset$. This means that $x \in \overline{V \cup W}$. Similarly, by the same argument, if $x \in \overline{W}$, we conclude that $x \in \overline{V \cup W}$.

For the second part of the question, we consider $(\mathbb{R}, \mathrm{d}_1)$, $\mathbb{Q}^\circ = \emptyset$ and $(\mathbb{R} \setminus \mathbb{Q})^\circ = \emptyset$, but $\mathbb{R}^\circ = \mathbb{R} \neq \emptyset$.

Exercise 6.6.* Let (X, d) be a metric space, and V be a subset of X. Prove that

(i) the set V° is open, and V° is the largest open set contained in V;

(ii) the set $\overline{V}$ is closed, and $\overline{V}$ is the smallest closed set which contains V.

Hint: For the latter part of (i), you need to show that if $\Omega \subseteq V$ and Ω is an open set in (X, d), then $\Omega \subseteq V^\circ$. For the latter part of (ii), you need to show that if $V \subseteq \Delta$ and Δ is a closed set in (X, d), then $\overline{V} \subseteq \Delta$.

Solution: (i) First we show that V° is an open set. Let z be an arbitrary point in V°. Then, there is $\delta > 0$ such that $B_\delta(z) \subset V$. We claim that $B_\delta(z) \subset V^\circ$. To see that, fix an arbitrary $y \in B_\delta(z)$. There is $r > 0$ such that $B_r(y) \subset B_\delta(z)$. Since, $B_\delta(z) \subset V$, we must have $B_r(y) \subset V$. This means that $y \in V^\circ$. As y in $B_\delta(z)$ was arbitrary, we conclude that $B_\delta(z) \subset V^\circ$. As $z \in V^\circ$ was arbitrary, we conclude that V° is an open set.

Now we show that V° is the largest open set contained in V. To see that, let Ω be an arbitrary open set contained in V. For every $z \in \Omega$, since Ω is an open set, there is $r > 0$ such that $B_r(z) \subset \Omega$. As $\Omega \subset V$, we have $B_r(z) \subset V$. This implies that $z \in V^\circ$. Since $z \in \Omega$ was arbitrary, we conclude that $\Omega \subset V^\circ$.

(ii) Let us first show that $\overline{V}$ is closed. To see that, let $(x_n)_{n\geq 1}$ be an arbitrary sequence in $\overline{V}$ which converges to some $x \in X$. Let $r > 0$ be arbitrary. Since $(x_n)_{n\geq 1}$ converges to x, there is $n \in \mathbb{N}$ such that $x_n \in B_{r/2}(x)$. Since $x_n \in \overline{V}$, the set $B_{r/2}(x_n) \cap V \neq \emptyset$, so there is $z \in B_{r/2}(x_n) \cap V$. Then,

$$\mathrm{d}(z, x) \leq \mathrm{d}(z, x_n) + \mathrm{d}(x_n, x) < r/2 + r/2 = r.$$

This implies that $z \in B_r(x)$, and hence $B_r(x) \cap V \neq \emptyset$. As $r > 0$ was arbitrary, we conclude that $x \in \overline{V}$. Since $(x_n)_{n\geq 1}$ was an arbitrary sequence in $\overline{V}$ (and we showed that its limit is contained in $\overline{V}$), we conclude that $\overline{V}$ is a closed set.

For the latter part of item (ii), assume that F is a closed set in X, which contains V. We need to show that $\overline{V} \subset F$. Let z be an arbitrary point in $\overline{V}$. By the definition of $\overline{V}$, for every $n \in \mathbb{N}$, there is $z_n \in B_{1/n}(z) \cap V$. This generates a sequence $(z_n)_{n\geq 1}$ in V which converges to z. Since $V \subset F$, the sequence $(z_n)_{n\geq 1}$ is contained in F, and because F is closed, we must have $z \in F$. Therefore, $\overline{V} \subset F$.

Exercise 6.7. Let (A_1, d_1) and (A_2, d_2) be metric spaces. A map $f : A_1 \to A_2$ is continuous if and only if the pre-image of any closed set in A_2 is a closed set in A_1.

Solution: Let $f : A_1 \to A_2$ be continuous, and a set $F \subset A_2$ be closed. Then $A_2 \setminus F$ is open, as a complement of an open set by a theorem in lectures. By another theorem in lectures, $f : A_1 \to A_2$ is continuous if and only if the preimage of any open set is open. Thus the preimage $f^{-1}(A_2 \setminus F)$ is open. But we know that the preimage of the complement is the complement of the preimage, $f^{-1}(A_2 \setminus F) = A_1 \setminus f^{-1}(F)$, so that $f^{-1}(F)$ is closed (as the complement of an open set).

Conversely, assume that the preimage of any closed set in A_2 is a closed set in A_1. Let $\Omega \subset A_2$ be open. Then $A_2 \setminus \Omega$ is closed. Therefore $f^{-1}(A_2 \setminus \Omega) = A_1 \setminus f^{-1}(\Omega)$ is closed. Hence $f^{-1}(\Omega)$ is open. Thus we showed that the preimage of any open set is open. Therefore, by a theorem in the lectures, f is continuous.

Exercise 6.8. Recall that the set of all continuous functions from $[0,1]$ to $\mathbb{R}$ is denoted by $C([0,1])$. We also defined the metrics d_1, d_2 and d_∞ on $C([0,1])$. Consider the map

$$\Phi : C([0,1]) \to \mathbb{R},$$

defined as

$$\Phi(f) = f(1/2).$$

(i) Is the map Φ from the metric space $(C([0,1]), \mathrm{d}_\infty)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

(ii) Is the map Φ from the metric space $(C([0,1]), \mathrm{d}_1)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

(iii) Is the map Φ from the metric space $(C([0,1]), \mathrm{d}_2)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

Hint: draw the graphs of few functions, and think about what it means for two functions in $C([0,1])$ to be close together in each of those metrics.

Solution: (i) Yes. To see this, let $\epsilon > 0$ be arbitrary. We define $\delta = \epsilon$. Assume that for some f and g in $C([a,b])$ we have

$$\mathrm{d}_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| < \delta.$$

Then,

$$\mathrm{d}_1(f(1/2), g(1/2)) = |f(1/2) - g(1/2)| \le \sup_{x\in[0,1]} |f(x) - g(x)| < \delta = \epsilon.$$

Since this holds for all f, g, the map Φ from the metric space $(C([0,1]), \mathrm{d}_\infty)$ to $(\mathbb{R}, \mathrm{d}_1)$ is continuous (indeed, it is uniformly continuous).

(ii) This is not continuous. To see that, consider the sequence of functions $(f_n)_{n\ge1}$ in $C([0,1])$ defined as follows. For each $n \ge 1$, let

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2 - 1/n], \\ nx - n/2 + 1 & \text{if } x \in [1/2 - 1/n, 1/2], \\ 1 - nx + n/2 & \text{if } x \in [1/2, 1/2 + 1/n], \\ 0 & \text{if } x \in [1/2 + 1/n, 1]. \end{cases}$$

Also consider the constant function $g \equiv 0$ on $[0,1]$. Then

$$\mathrm{d}_1(f_n, g) = \int_0^1 |f_n(t) - g(t)| dt = \frac{1}{n}.$$

Thus, $\mathrm{d}(f_n, g) \to 0$ as $n \to \infty$. Therefore, f_n converges to g in the metric space $(C([0,1]), \mathrm{d}_1)$. However, by construction $\Phi(f_n) = f_n(1/2) = 1$ for all n, so $\Phi(f_n)$ converges to 1 in $(\mathbb{R}, \mathrm{d}_1)$. But, $\Phi(g) = g(1/2) = 0$. Therefore, $\Phi(f_n)$ does not converge to $\Phi(g)$ as $n \to \infty$.

(iii) This is not continuous. The example in part (ii) works in this case as well.

Exercise 6.9. Consider the metric spaces $X = (\mathbb{R}, \mathrm{d}_1)$ and $Y = (\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$. Show that the map $f(x) = x$ from X to Y is not continuous. Show that the map $g(x) = x$ from Y to X is continuous.

Solution: Recall that in the discrete metric, any set is open. Also, a map is continuous iff the preimage of any open set is an open set. Consider the open set $[0, 1]$ in Y. The preimage of this set under f is $[0, 1]$, which is not open in $X = (\mathbb{R}, \mathrm{d}_1)$. Therefore, $f : X \to Y$ is not continuous.

Take any open set A in X. Its preimage $g^{-1}(A) = A$ is a subset of Y and therefore it is open in Y. Hence, $g : Y \to X$ is continuous.

Exercise 6.10. Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$, for $n \geq 1$, defined as

$$f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : [0, 1] \to \mathbb{R}$ be the constant map $f \equiv 0$.

(i) Show that the sequence $(f_n)_{n\geq1}$ in $C([0, 1])$ converges to f in the metric space $(C([0, 1], \mathrm{d}_1)$.

(ii) Show that the sequence $(f_n)_{n\geq1}$ in $C([0, 1])$ does not converge to f in the metric space $(C([0, 1], \mathrm{d}_\infty)$.

(iii) Conclude that the identity map

$$\mathrm{id} : (C([0, 1]), \mathrm{d}_1) \to (C([0, 1]), \mathrm{d}_\infty)$$

is not continuous.

Solution: (i) We have

$$\mathrm{d}_1(f_n, 0) = \int_0^1 |f_n(t)| dt = \frac{1}{2n} \to 0, \qquad \text{as } n \to \infty.$$

This implies that $(f_n)_{n\geq1}$ converges to $f \equiv 0$ in $(C([0, 1], \mathrm{d}_1)$.

(ii) We have

$$\mathrm{d}_\infty(f_n, 0) = \sup_{x\in[0,1]} |f_n(x)| = 1, \qquad \forall n \geq 1.$$

Therefore, $\mathrm{d}_\infty(f_n, 0)$ does not tend to zero as $n \to \infty$. Thus, $(f_n)_{n\geq1}$ does not converge to $f \equiv 0$ in the metric space $(C([0, 1], \mathrm{d}_\infty)$.

(iii) We have $\mathrm{id}(f_n) = f_n \in (C([0, 1], \mathrm{d}_\infty)$. In part (i) we showed that $f_n \to f$ in $(C([0, 1], \mathrm{d}_1)$. If id is continuous, we must have $\mathrm{id}(f_n) \to \mathrm{id}(0) = f$ as $n \to \infty$, in the metric space $(C([0, 1], \mathrm{d}_\infty)$. However, in part (ii) we showed that $(f_n)_{n\geq1} \in (C([0, 1], \mathrm{d}_\infty)$ does not converge to f in the metric space $(C([0, 1], \mathrm{d}_\infty)$. This contradiction shows that id is not continuous.

Exercise 6.11. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a surjective map. Show that if f is bi-Liptschitz, then it is a homeomorphisms.

Solution: We have some constants $M_1, M_2 > 0$ s.t. for all $x, y \in X$

$$M_1 \,\mathrm{d}_X(x,y) \leq \mathrm{d}_Y(f(x), f(y)) \leq M_2 \,\mathrm{d}_X(x,y).$$

Let us first show that f is injective. Assume that for some x_1 and x_2 in X we have $f(x_1) = f(x_2)$. Then, by the above inequality, we must have $\mathrm{d}_X(x,y) \leq 0$. By the properties of the metric d_X, that implies that $x = y$. Therefore, f is injective.

By the hypothesis f is surjective. Therefore, f is a bijective map.

Let us first show that f is continuous. Fix an arbitrary $\epsilon > 0$. Let $\delta = \epsilon/M_2$. For every x_1 and x_2 in X with $\mathrm{d}_X(x_1, x_2) < \delta$, we have $\mathrm{d}_Y(f(x_1), f(x_2)) \leq M_2 \,\mathrm{d}_X(x,y) < \epsilon$. Therefore, f is continuous (indeed, it is uniformly continuous).

Now we show that f^{-1} is continuous. Let $\epsilon > 0$ be arbitrary. Define $\delta = \epsilon M_1$. Let y_1 and y_2 be arbitrary elements in Y such that $\mathrm{d}_Y(y_1, y_2) < \delta$. Then,

$$\mathrm{d}_X(f^{-1}(y_1), f^{-1}(y_2)) \leq \frac{1}{M_1} \mathrm{d}_Y(y_1, y_2) < \frac{1}{M_1} M_1 \epsilon = \epsilon.$$