Exercise 6.1. Let $(X, \mathrm{d}_{\mathrm{disc}})$ be a discrete metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Then, $(x_n)_{n\geq 1}$ converges in $(X, \mathrm{d}_{\mathrm{disc}})$ if and only if the sequence $(x_n)_{n\geq 1}$ is eventually constant.

Exercise 6.2. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Prove that the sequence $(x_n)_{n\geq 1}$ converges to $x \in X$ if and only if, for every open set U in (X, d) with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.

Exercise 6.3. Let $(X, \mathrm{d}_{\mathrm{disc}})$ be a discrete metric space. Then every set in X is closed.

Exercise 6.4. Let (X, d) be a metric space, and V be a subset of X. Show that the set V is closed if and only if $\overline{V} = V$.

Exercise 6.5. Let V and W be subsets of a metric space (X, d). Prove that

$$\overline{V \cup W} = \overline{V} \cup \overline{W}.$$

Give an example of (X, d), V and W such that

$$(V \cup W)^\circ \neq V^\circ \cup W^\circ.$$

Exercise 6.6.* Let (X, d) be a metric space, and V be a subset of X. Prove that

(i) the set V° is open, and V° is the largest open set contained in V;

(ii) the set $\overline{V}$ is closed, and $\overline{V}$ is the smallest closed set which contains V.

Exercise 6.7. Let (A_1, d_1) and (A_2, d_2) be metric spaces. A map $f : A_1 \to A_2$ is continuous if and only if the pre-image of any closed set in A_2 is a closed set in A_1.

Exercise 6.8. Recall that the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$ is denoted by $C([0, 1])$. We also defined the metrics d_1, d_2 and d_∞ on $C([0, 1])$. Consider the map

$$\Phi : C([0, 1]) \to \mathbb{R},$$

defined as

$$\Phi(f) = f(1/2).$$

(i) Is the map Φ from the metric space $(C([0, 1]), \mathrm{d}_\infty)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

(ii) Is the map Φ from the metric space $(C([0, 1]), \mathrm{d}_1)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

(iii) Is the map Φ from the metric space $(C([0, 1]), \mathrm{d}_2)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous? Justify your answer.

Exercise 6.9. Consider the metric spaces $X = (\mathbb{R}, \mathrm{d}_1)$ and $Y = (\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$. Show that the map $f(x) = x$ from X to Y is not continuous. Show that the map $g(x) = x$ from Y to X is continuous.

Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional

Exercise 6.10. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$, for $n \geq 1$, defined as

$$f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : [0,1] \to \mathbb{R}$ be the constant map $f \equiv 0$.

(i) Show that the sequence $(f_n)_{n\geq 1}$ in $C([0,1])$ converges to f in the metric space $(C([0,1], \mathrm{d}_1)$.

(ii) Show that the sequence $(f_n)_{n\geq 1}$ in $C([0,1])$ does not converge to f in the metric space $(C([0,1], \mathrm{d}_\infty)$.

(iii) Conclude that the identity map

$$\mathrm{id} : (C([0,1]), \mathrm{d}_1) \to (C([0,1]), \mathrm{d}_\infty)$$

is not continuous.

Exercise 6.11. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a surjective map. Show that if f is bi-Liptschitz, then it is a homeomorphisms.