Exercise 8.1. Let (X, d) be a metric space. Show that X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$.

Hint: In one direction, you have a pair of separating sets, and you can consider of the open sets in the pair. In the other direction, consider the particular set and its complement.

Solution: First assume that X is connected. Let U be an arbitrary subset of X which is both open and closed. We need to show that either $U = X$ or $U = \emptyset$. Consider the set $V = X \setminus U$. Since U is closed in X, V is open in X. We also have $U \cap V = \emptyset$, and $U \cup V = X$. If both U and V are not empty, then U and V disconnect X, which contradicts the assumption. Therefore, at least one of U and V is empty. This implies that either $U = \emptyset$ or $U = X$.

Now, assume that the only subsets of X which are both open and closed are X and $\emptyset$. Assume in the contrary that X is not connected. Then, there exist $U, V \subset X$ such that U and V are open in X, are not empty, are disjoint, and $X = U \cup V$. Therefore $V = X \setminus U$ is closed (complement of the open set U). These imply that $V \neq X$ and $V \neq \emptyset$, which is a contradiction.

Exercise 8.2. Show that in the Euclidean metric space $(\mathbb{R}^1, \mathrm{d}_1)$, the set of rational numbers $\mathbb{Q}$ is disconnected.

Hint:pick an irrational number, and consider the set of rational numbers less than that number, and the set of rational numbers larger than that set.

Solution: Consider the sets

$$U = (-\infty, \sqrt{2}) \qquad V = (\sqrt{2}, +\infty).$$

Then U and V are open in $\mathbb{R}$, and we have

$$\mathbb{Q} \subset U \cup V, \quad U \cap V = \emptyset, \quad U \neq \emptyset, \quad V \neq \emptyset.$$

These show that U and V disconnect Q.

Exercise 8.3.* Consider the Euclidean metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that a and b are real numbers with $a < b$.

(i) Show that the interval $[a, b)$ is connected.

Hint: This is a special case of the proof of the connectivity of $[a, b]$

(ii) Show that the interval $(a, b]$ is connected.

Hint: Modify the proof of the thm showing that $[a, b$ *is connected; starting with* b *instead of* a*, modify* I*, and take the infimum of* I*.*

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Questions marked with $*$ are optional

(iii) Show that the interval (a, b) is connected.

Hint: Choose $u \in U \cap (a, b)$ and $v \in V \cap (a, b)$, and consider the interval $[u, v]$ or $[v, u]$, depending on $u < v$ or $v < u$.

Solution: This is a special case of the proof of the connectedness of the interval $[a, b]$. Assume in the contrary that $[a, b)$ is not connected. Then, there are open sets U and V in $\mathbb{R}$ such that

$$U \cap V = \emptyset, \quad U \cap [a, b) \neq \emptyset, \quad V \cap [a, b) \neq \emptyset, \quad [a, b) \subseteq U \cup V.$$

We assume without loss of generality that $a \in U$ (otherwise exchange the names of U, V). Consider the set

$$I = \{s \in [a, b) \mid [a, s] \subset U\}.$$

Let $t = \sup I$. We consider three cases based on the value of t.

(1) Assume that $t = a$. As U is open there is $\delta > 0$ such that $[a - \delta, a + \delta) \in U$. Since $a < b$, we may make δ smaller so that $a + \delta < b$. Therefore, $[a, a + \delta/2] \subset [a, b)$, and hence $t > a$. This contradiction shows that this case cannot occur.

(2) Assume that $t = b$. Because t is the supremum of I, for every $s_1 \in [a, b)$, there is $s > s_1$ such that $s \in I$. Hence, $s_1 \in [a, s] \subset U$. This shows that $[a, b) = [a, t) \subset U$. Then, $V \cap [a, b) = \emptyset$, which is not possible.

(3) $a < t < b$. As in the previous case, we note that $[a, t) \subset U$. If $t \in U$, there is $\epsilon > 0$ such that $(t - \epsilon, t + \epsilon) \subset U$ and $t + \epsilon < b$. This contradicts $t = \sup I$. If $t \in V$, there is $\epsilon' > 0$ such that $(t - \epsilon', t + \epsilon') \in V$, which contradicts that U and V are disjoint.

In all possibilities for the value of t we reached a contradiction. Therefore, there cannot be U and V satisfying the above properties.

(ii): This is similar to case (i); assume that $b \in U$, and consider the set

$$I = \{s \in (a, b] \mid [s, b] \subset U\},$$

let $t = \inf I$, and repeat the same argument.

(iii) Suppose for contradiction (a, b) is not connected. Then, there are open sets U and V such that

$$(a, b) \subset U \cup V, \quad U \cap V = \emptyset, \quad U \cap (a, b) \neq \emptyset, \quad V \cap (a, b) \neq \emptyset.$$

Choose $u \in U \cap (a, b)$ and $v \in V \cap (a, b)$. Assume without loss of generality that $u < v$. Now,

$$[u, v] \subset U \cup V, \quad U \cap V = \emptyset, \quad U \cap [u, v] \neq \emptyset, \quad V \cap [u, v] \neq \emptyset.$$

These imply that $[u, v]$ is disconnected. But in the lectures we have proved that any set of the from $[u, v]$ is connected.

Exercise 8.4. Show that the following metric spaces are path connected.

(i) the Euclidean space $\mathbb{R}^n$, for any $n \geq 1$,

(ii) the open ball $B_1(0)$ in $(\mathbb{R}^n, \mathrm{d}_2)$, for any $n \geq 2$,

(iii) the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq \|(x, y)\| \leq 2\}$.

Hint: For items (i) and (ii), consider a straight line segment between any pair of points. For item (iii), write an explicit formula for a curve spiralling from x to y, using the polar coordinates.

Solution: (i): Let x and y be arbitrary elements in $\mathbb{R}^n$. Consider the map $f:[0,1]\to\mathbb{R}^n$, defined as

$$f(t)=(1-t)x+ty, \qquad \text{for } t\in[0,1].$$

We have $f(0)=x$ and $f(1)=y$. We need to show that f is continuous. If $x=y$, then f is a constant map, and hence is continuous. If $x\neq y$, for every $\epsilon>0$, we consider $\delta=\epsilon/\|x-y\|$, and note that for every s and t in $[0,1]$ satisfying $|s-t|<\delta$ we have

$$\begin{aligned}\|f(s)-f(t)\| &= \|((1-s)x+sy)-((1-t)x+ty)\|\\ &= |t-s|\|x-y\|\\ &< \frac{\epsilon}{\|x-y\|}\|x-y\|\\ &= \epsilon.\end{aligned}$$

Thus, f is continuous. As x and y are arbitrary, we conclude that f is continuous.

(ii): Let x and y in $B_1(0)$ be arbitrary elements. We consider the map $f:[0,1]\to\mathbb{R}$ defined as

$$f(t)=(1-t)x+ty, \qquad \text{for } t\in[0,1].$$

We note that for every $t\in[0,1]$, we have

$$\|f(t)\|=\|(1-t)x+ty\|\leq(1-t)\|x\|+t\|y\|<(1-t)+t=1.$$

This shows that for every $t\in[0,1]$, $f(t)\in B_1(0)$. In other words, $f:[0,1]\to B_1(0)$. We already proved in part (i) that f is continuous. This shows that $B_1(0)$ is path connected.

(iii) Let x_1 and x_2 be arbitrary elements in

$$\{z\in\mathbb{R}^2 \mid 1\leq\|z\|\leq 2\}.$$

There are $\theta_1\in[0,2\pi)$ and $\theta_2\in[0,2\pi)$ such that

$$x_1=\|x_1\|\,(\cos\theta_1,\sin\theta_1), \quad x_2=\|x_2\|\,(\cos\theta_2,\sin\theta_2)$$

Let us consider the map $f:[0,1]\to\mathbb{R}^2$, defined as

$$f(t)=\Big((1-t)\|x_1\|+t\|x_2\|\Big)\Big(\cos((1-t)\theta_1+t\theta_2),\sin((1-t)\theta_1+t\theta_2)\Big).$$

We have $f(0)=x_1$ and $f(1)=x_2$. Since sin and cos are continuous functions, the map f is continuous. We need to show that f is a map from $[0,1]$ to $\{z\in\mathbb{R}^2 \mid 1\leq\|z\|\leq 2\}$. For every $t\in[0,1]$, since x_1 and x_2 satisfy $1\leq\|x_1\|\leq 2$ and $1\leq\|x_2\|\leq 2$, we have

$$\|f(t)\|=(1-t)\|x_1\|+t\|x_2\|\in[1,2].$$

Exercise 8.5. Consider the set of all continuous functions $f:[0,1]\to\mathbb{R}$, that is $C([0,1])$, with the metric d_1.

(i) Show that the space $(C([0,1]), \mathrm{d}_1)$ is path connected.

(ii) Conclude that the space $(C([0,1]), \mathrm{d}_1)$ is connected.

Hint: For arbitrary f and g in $C([0,1])$, define an explicit map $\phi : [0,1] \to C([0,1])$ defined as a linear combination of f and g. You need to show that every such linear combination belongs to $C([0,1])$, and the map Φ is continuous with respect to d_1.

Solution: Let f and g be arbitrary elements in $C([0,1])$. Consider the map $\Phi : [0,1] \to C([0,1])$, defined as

$$\Phi(t) = (1-t)f + tg.$$

Obviously, for every $t \in [0,1]$, $(1-t)f + tg$ is a continuous function on $[0,1]$. Therefore, Φ maps into $C([0,1])$. We also have $\Phi(0) = f$ and $\Phi(1) = g$. We need to show that $\Phi(t)$ is continuous. If $f = g$ (that is $f(x) = g(x)$ for all $x \in [0,1]$), then Φ is a constant map, and hence it is continuous. So let us assume that $f \neq g$ (there is $x \in [0,1]$ such that $f(x) \neq g(x)$). For $\epsilon > 0$, let $\delta = \epsilon / \mathrm{d}_1(f,g)$. For every s and t in $[0,1]$ with $|s-t| < \delta$, we have

$$\begin{aligned}
\mathrm{d}_1\big(\Phi(s), \Phi(t)\big) &= \int_0^1 |\Phi(s) - \Phi(t)|\, dx \\
&= \int_0^1 \big|\big((1-s)f + sg\big) - \big((1-t)f + tg\big)\big|\, dx \\
&= |s-t| \int_0^1 |f-g|\, dx = |s-t|\, \mathrm{d}_1(f,g) \\
&\leq \frac{\epsilon}{\mathrm{d}_1(f,g)}\, \mathrm{d}_1(f,g) = \epsilon.
\end{aligned}$$

This shows that Φ is continuous on $[0,1]$.

Part (ii) of the problem follows from the theorem in the lectures that every path connected metric space is connected.

Exercise 8.6.* In this exercise, we aim to show that a connected space may not be path connected.

Consider the following subset of $\mathbb{R}^2$:

$$A = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\} \cup \{(x,y) \in \mathbb{R}^2 \mid x = 0, y \in [-1,+1]\}.$$

That is, A is the union of the oscillating curve which is the graph of $\sin(1/x)$, and the vertical line segment $\{0\} \times [-1,+1]$.

(i) show that the set A is connected.

Hint: first show that each of the vertical line segment and the graph of $\sin(1/x)$ are connected. So the only way to disconnect A is to separate those two pieces by open sets. However, any open set containing the straight line segment, will also contain part of the graph.

(ii) show that the set A is not path connected.

Hint: You need to show that there is no path joining a point on the line segment to a point on the graph.

Solution: (i): In order to show that A is connected, by a theorem in the lectures, it is enough to show that there is no continuous and surjective map from A to $\{0,1\}$. To see that, let $f : A \to \{0,1\}$ be a continuous map. We aim to show that f cannot be surjecitve.

Consider the sets

$$A_1 = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\}, \qquad A_2 = \{(x,y) \in \mathbb{R}^2 \mid x = 0, y \in [-1,+1]\}$$

The set A_1 is connected. That is because, it is homeomorphic to the set $(0,+\infty)$, and the set $(0,+\infty)$ is connected (the proof is similar to the arguments in Exercise 8.3-(iii)). Since $f : A_1 \to \{0,1\}$ is continuous, and A_1 is connected, by a theorem in the lectures, f cannot be surjective. Thus, wither $f(A_1) = 0$ or $f(A_1) = 1$. Let us assume that $f(A_1) = 0$ (the other case is similar).

Let $(0,y)$ be an arbitrary point in A_2. There is a sequence of points $(x_n)_{n\geq 0}$ in $(0,+\infty)$ such that $(x_n, \sin(x_n^{-1})) \to (0,y)$, as $n \to \infty$. Since f is continuous on A, we conclude that

$$f(0,y) = \lim_{n\to\infty} f\left(x_n, \sin(1/x_n)\right) = \lim_{n\to\infty} 0 = 0.$$

Since $(0,y)$ in A_2 was arbitrary, we conclude that $f(A_2) = 0$. Combining with the previous paragraph, we conclude that $f(A) \equiv 0$, thus, f cannot be surjective.

(ii) Assume in the contrary that A is path connected. There must be a continuous map

$$f : [0,1] \to A$$

such that

$$f(0) = (0,0), \qquad f(1) = (1, \sin(1)).$$

Since f is continuous, and A_2 is closed, $I = f^{-1}(A_2) \subset [0,1]$ is a closed set. Define $t = \sup I$. Since I is closed, $t \in I$. It follows that $f(t) \in A_2$, and for all $s \in (t,1]$, $f(s) \in A_1$. In particular, $t < 1$. Below we aim to show that f cannot be continuous at t, since its limit as x tends to t from the right hand side does not exist (due to the oscillations).

Let us write the map $f : [0,1] \to A$ in its coordinates

$$f(x) = (f^1(x), f^2(x)),$$

for some continuous functions $f^1 : [0,1] \to \mathbb{R}$ and $f^2 : [0,1] \to \mathbb{R}$. Since f^2 is continuous, for $\epsilon = 1/4$, there is $\delta > 0$ such that

$$\forall s \in [t, t+\delta], |f^2(s) - f^2(t)| \leq 1/4.$$

This implies that

$$f^2([t,t+\delta]) \neq [-1,+1].$$

Since f^1 is continuous on $[t,t+\delta]$ and $[t,t+\delta]$ is connected, $f^1([t,t+\delta])$ is connected. By a theorem in the lectures, $f^1([t,t+\delta])$ is an interval. On the other hand, by the first paragraph, we have $f(t+\delta) \in A_2$, which implies that $f^1(t+\delta) > 0$. We also have $f^1(t) = 0$. Therefore,

$$[0, f^1(t+\delta)] \subset f^1([t,t+\delta]).$$

Let us choose $k \in \mathbb{N}$ such that

$$\left[\frac{1}{2k\pi - \pi/2}, \frac{1}{2k\pi + \pi/2}\right] \subseteq [0, f^1(t+\delta)].$$

Then, by the previous inclusion, there is a set $S \subset [t, t+\delta]$ such that

$$f^1(S) = \left[\frac{1}{2k\pi - \pi/2}, \frac{1}{2k\pi + \pi/2}\right].$$

This implies that $f^2(S) = [-1, 1]$. However, this contradicts $f^2([t, t+\delta]) \neq [-1, +1]$.

Unseen Exercise. (unseen) The purpose of this exercise is to give a direct proof that a path connected space is connected.

Let us assume that there is a metric space (X, d) which is path connected, but not connected. By the definition of connected sets, there must be open sets U and V in X such that $X = U \cup V$, $U \cap V = \emptyset$, $U \neq \emptyset$, and $V \neq \emptyset$.

Let us choose a point $u \in U$ and a point $v \in V$ (we can do this since U and V are not empty.). Since X is path connected, there is a continuous map $g : [0, 1] \to X$ satisfying $g(0) = u$ and $g(1) = v$. Show that the sets

$$U' = g^{-1}(U), \qquad V' = g^{-1}(V),$$

disconnect $[0, 1]$.

Solution: Since g is continuous, both U' and V' are open sets in $[0, 1]$ (pre-images of open sets by a continuous map). As $U \cup V = X$, $U' \cup V' = [0, 1]$. As $f(0) = u \in U$, $U' \neq \emptyset$, and as $v = f(1) \in V$, $V' \neq \emptyset$. Also, since $U \cap V = \emptyset$, $U' \cap V' = \emptyset$. These show that the metric space $([0, 1], \mathrm{d}_1)$ is disconnected, where d_1 is the induced metric from d_1 on $\mathbb{R}$.