Exercise 8.1. Let (X, d) be a metric space. Show that X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$.

Exercise 8.2. Show that in the Euclidean metric space $(\mathbb{R}^1, \mathrm{d}_1)$, the set of rational numbers $\mathbb{Q}$ is disconnected.

Exercise 8.3.* Consider the Euclidean metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that a and b are real numbers with $a < b$.

(i) Show that the interval $[a, b)$ is connected.

(ii) Show that the interval $(a, b]$ is connected.

(iii) Show that the interval (a, b) is connected.

Exercise 8.4. Show that the following metric spaces are path connected.

(i) the Euclidean space $\mathbb{R}^n$, for any $n \geq 1$,

(ii) the open ball $B_1(0)$ in $(\mathbb{R}^n, \mathrm{d}_2)$, for any $n \geq 2$,

(iii) the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq \|(x, y)\| \leq 2\}$.

Exercise 8.5. Consider the set of all continuous functions $f : [0, 1] \to \mathbb{R}$, that is $C([0, 1])$, with the metric d_1.

(i) Show that the space $(C([0, 1]), \mathrm{d}_1)$ is path connected.

(ii) Conclude that the space $(C([0, 1]), \mathrm{d}_1)$ is connected.

Exercise 8.6. * In this exercise, we aim to show that a connected space may not be path connected.

Consider the following subset of $\mathbb{R}^2$:

$$A = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid x = 0, y \in [-1, +1]\}.$$

That is, A is the union of the oscillating curve which is the graph of $\sin(1/x)$, and the vertical line segment $\{0\} \times [-1, +1]$.

(i) show that the set A is connected.

(ii) show that the set A is not path connected.

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Questions marked with $*$ are optional