**Exercise 9.1.** Consider the metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that $a$ and $b$ are real numbers with $a < b$. Show that all of the intervals $(a, b]$, $[a, b)$, $[a, +\infty)$, and $(-\infty, b]$ are not compact.

**Exercise 9.2.** Show that if $A$ and $B$ are compact subsets of a metric space $(X, \mathrm{d})$, then $A \cup B$ is a compact set.

**Exercise 9.3.** Show that the ball

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

in the metric space $(\mathbb{R}^2, \mathrm{d}_2)$ is not compact.

**Exercise 9.4.** Let $(X, \mathrm{d})$ be a metric space, and $A_1$, $A_2$, $\dots$, $A_n$ be a finite number of bounded sets in $X$. Then $\cup_{i=1}^n A_i$ is a bounded set in $X$.

**Exercise 9.5.** Let $(X, \mathrm{d})$ be a non-empty metric space, and let $Z \subseteq X$. Show that $Z$ is bounded if and only if there is $x \in X$ and $r \in \mathbb{R}$ such that $Z \subseteq B_r(x)$.

**Exercise 9.6.** Consider the set $\mathbb{R}$ with the discrete metric $\mathrm{d}_{\mathrm{disc}}$. The set $(0, 1)$ is closed and bounded in $(\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$, but it is not compact.

**Exercise 9.7.** Let $(X, \mathrm{d})$ be a metric space, and assume that $V_n$, for $n \geq 1$, be a nest of non-empty closed sets in $X$.

(i) Show that if $X$ is compact, then $\cap_{n\geq 1} V_n$ is not empty.

(ii) Give an example of a nest of non-empty closed sets $V_n$, for $n \geq 1$, in a metric space such that $\cap_{n\geq 1} V_n$ is empty.

**Exercise 9.8.** Show that if a metric space is sequentially compact, then it is bounded.

**Exercise 9.9.*** Let $(X, \mathrm{d})$ be a sequentially compact metric space. Show that $X$ is separable, that is, there is a countable dense set in $X$.

**Exercise 9.10.*** Let $(X, \mathrm{d})$ be a sequentially compact metric space, and $\mathcal{R}$ be an open cover for $X$. Show that there is a countable sub-cover of $\mathcal{R}$ for $X$.

**Exercise 9.11.** Let $(X, \mathrm{d})$ be a compact metric space, and assume that $f : X \to X$ is a continuous map such that for all $x \in X$, we have $f(x) \neq x$. Show that there is $\delta > 0$ such that for all $x \in X$, we have $\mathrm{d}(x, f(x)) \geq \delta$.

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Please send any corrections to d.cheraghi@imperial.ac.uk
Questions marked with $*$ are optional