MATH50001 Problems Sheet 1
Solutions

1a. Let $z = x + iy$. Then $iz = ix - y$ which implies $\operatorname{Re}(iz) = -y = -\operatorname{Im} z$ and $\operatorname{Im}(iz) = \operatorname{Re} z$.

2. a)

$$\frac{3+2i}{1+i} = \frac{3+2i}{1+i} \cdot \frac{1-i}{1-i} = \frac{3+2+i(2-3)}{2} = \frac{5}{2} - i\,\frac{1}{2}.$$

b)

$$\frac{1+i}{3-i} = \frac{1+i}{3-i} \cdot \frac{3+i}{3+i} = \frac{3-1+i(3+1)}{10} = \frac{1}{5} + i\,\frac{2}{5}.$$

c)

$$\frac{z+2}{z+1} = \frac{(x+2)+iy}{(x+1)+iy} = \frac{(x+2)+iy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x+2)(x+1)+y^2+i\left((x+1)y-(x+2)y\right)}{(x+1)^2+y^2}$$

$$= \frac{x^2+3x+2+y^2-i\,y}{(x+1)^2+y^2} = \frac{x^2+3x+2+y^2}{(x+1)^2+y^2} - i\,\frac{y}{(x+1)^2+y^2}.$$

3. $z = x + iy = r(\cos\theta + i\sin\theta)$, where

$r = \sqrt{x^2+y^2}, \qquad \operatorname{Arg} z = \theta = \arcsin(y/r) = \arccos x/r, \quad \text{s.t.} \quad -\pi < \theta \le \pi.$

a) $|z| = r = \sqrt{2}$, $\operatorname{Arg} z = \arcsin(-1/\sqrt{2}) = -\pi/4$.

b) $|z| = r = 3$, $\operatorname{Arg} z = \arcsin(-1) = -\pi/2$.

c) $|z| = r = 5$, $\operatorname{Arg} z = \arcsin(4/5)$.

d) $|z| = r = \sqrt{5}$, $\operatorname{Arg} z = -\arccos(1/\sqrt{5})$.

4. $|z| = r = \sqrt{2}$, $\operatorname{Arg} z = \arcsin(1/\sqrt{2}) = \pi/4$.

$$(1+i)^{16} = r^{16}(\cos(16\cdot\pi/4) + i\sin(16\cdot\pi/4)) = 256.$$

5.

a) We can assume that z_1 and z_2 are two points on the real line, so that $z_1 = x_1$ and $z_2 = x_2$. Then

$$\{z : |z - z_1| = |z - z_2|\} = \left\{z : z = \frac{x_1+x_2}{2} + iy,\ y \in \mathbb{R}\right\}.$$

b) The equation $1/z = \bar{z}$ iff $|z|^2 = 1$ that is the unit circle.

c) $\operatorname{Re} z = 3$ coincide with the straight line $\{z = x + iy\, x = 3\}$.

d) $\operatorname{Re} z = x + iy > c$ is a half plane whose $x > c$.

e) Let $z = x+iy$, $a = a_1 + ia_2$ and $b = b_1 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and let $a \neq 0$. Note that

$$\operatorname{Re} za + b = xa_1 - ya_2 + b_1 > 0$$

If $a_2 = 0$ then $x > -b_1/a_1$ and if $a_2 \neq 0$ then $y < \frac{a_1}{a_2} x + b_1$.

f) Let $z = x + iy$. Then we have $\sqrt{x^2 + y^2} = x + 1$. This implies that

$$y^2 = 2x + 1 \quad \text{with} \quad x \geq -1.$$

g) Let $z = x + iy$. Then $\operatorname{Im} z = y = c$ is the equation of a line parallel to the real axis.

6.

$$< z_1, z_2 >= x_1x_2 + y_1y_2.$$

$$\frac{1}{2}\left[(z_1, z_2) + (z_2, z_1)\right] = \frac{1}{2}\left[z_1\bar{z}_2 + z_2\bar{z}_1\right]$$

$$= \frac{1}{2}\Big(x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2) + x_1x_2 + y_1y_2 + i(x_1y_2 - x_2y_1)\Big)$$

$$= x_1x_2 + y_1y_2.$$

7. Here $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Since $u'_x = 2x$ and $u'_y = 2y$, the Cauchy-Riemann equations are satisfied only at $z = 0$. Hence differentiability fails at all non-zero points.

To verify differentiability at 0, we observe that

$$\frac{|z|^2 - 0}{z - 0} = \frac{z\bar{z}}{z} = \bar{z} \to 0$$

as $z \to 0$. Then the derivative exists and has value 0.

8. If $f(x+iy) = u(x, y) + iv(x, y) = \sqrt{|x||y|}$, then $u(x, y) = \sqrt{|x||y|}$ and $v(x, y) = 0$.

Since the function $u(x, y)$ takes the constant value 0 along both the x- and the y-axes we conclude that $u'_x = u'_y = 0$.

More formally: at the point (0, 0)

$$u'_x = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x \to 0} \frac{0}{x} = 0,$$

and the computation for u'_y is essentially identical. Thus the Cauchy- Riemann equations are trivially satisfied. On the other hand

$$\frac{f(z)-f(0)}{z-0} = \frac{\sqrt{|x||y|}}{x+iy} = \frac{\sqrt{\cos\theta\ \sin\theta}}{\cos\theta + i\sin\theta}, \qquad (*)$$

where $x = r\cos\theta$ and $y = r\sin\theta$. Thus $(*)$ is independent of r and if we take $\theta = 0$ or $= \pi/2$ then $(f(z)-f(0))/(z-0) = 0$ but if $\theta = \pi/4$ then $(f(z)-f(0))/(z-0) = (1-i)/2$.
We are forced to conclude that the limit does not exist as $z \to 0$.

9. Consider

$$\sum_{k=0}^{n-1} e^{2ik\pi/n} = 1 + e^{2i\pi/n} + e^{4i\pi/n} + \cdots + e^{2(n-1)i\pi/n} = \frac{1-e^{2i\pi}}{1-e^{2i\pi/n}} = 0. \quad (*)$$

Note that

$$e^{2ik\pi/n} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

The complex number equals to zero iff its real and imaginary parts are zeros. Therefore $(*)$ implies

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cdots + \cos\left(\frac{2(n-1)\pi}{n}\right) = -1$$

and

$$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \cdots + \sin\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

10).

(i)
If $C \neq 0$, then the images of the straight lines $x = C$ are circles

$$u^2 + v^2 - \frac{u}{C} = 0.$$

If $C = 0$, then the image is the axis $u = 0$.

If $C \neq 0$, then the images of the straight lines $y = C$ are circles $u^2 + v^2 + \frac{v}{C}$.
If $C = 0$, then the image is the axis $v = 0$.

(ii) The images of the circles $|z| = R$ are the circles $|w| = 1/R$.

(iii) The images of the rays $\arg z = \alpha$ are the rays $\arg w = -\alpha$.

(iv) The image of the circle $|z-1| = 1$ is the straight line $u = 1/2$.