MATH50001 - Problems Sheet 1

1. Show that Re $(iz) = -$Im z and Im $(iz) =$ Re z.

2. Write each of the following complex numbers in the standard form $a+bi$, where $a, b \in \mathbb{R}$

a) $(3+2i)/(1+i)$,
b) $(1+i)/(3-i)$,
c) $(z+2)/(z+1)$, where $z = x+iy$ with $x, y \in \mathbb{R}$.

3. Calculate the modulus and principal argument of

a) $1-i$; b) $-3i$; c) $3+4i$; d) $-1+2i$.

4. Express $z = 1+i$ in polar form and then calculate $(1+i)^{16}$.

5. Describe geometrically the sets of points z in the complex plane defined by the following relations:

a) $|z - z_1| = |z - z_2|$, where $z_1, z_2 \in \mathbb{C}$;
b) $1/z = \bar{z}$;
c) Re $(z) = 3$;
d) Re $(z) > c$, (resp., $\geq c$) where $c \in \mathbb{R}$;
e) Re $(az+b) > 0$ where $a, b \in \mathbb{C}$;
f) $|z| =$ Re $(z) + 1$;
g) Im $(z) = c$ with $c \in \mathbb{R}$.

6. Let $< \cdot, \cdot >$ denote the usual inner product in $\mathbb{R}^2$. In other words, if $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$, then

$$< Z_1, Z_2 >= x_1 x_2 + y_1 y_2.$$

Similarly, we may define a Hermitian inner product $(\cdot, \cdot)$ in $\mathbb{C}$ by

$$(z_1, z_2) = z_1 \bar{z}_2.$$

The term Hermitian is used to describe the fact that$(\cdot, \cdot)$ is not symmetric, but rather satisfies the relation

$$(z_1, z_2) = \overline{(z_2, z_1)} \quad \forall z_1, z_2 \in \mathbb{C}.$$

Show that

$$< z_1, z_2 >= \frac{1}{2}[(z_1, z_2) + (z_2, z_1)] = \text{Re}\,(z_1, z_2),$$

where we use the usual identification $z = x+iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

7. Show that the function $z \longrightarrow |z|^2$ is differentiable only at 0.

8. Consider the function defined by

$$f(x+iy) = \sqrt{|x||y|}, \qquad x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

9. Let n be an integer greater of equal than 2. Prove that

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cdots + \cos\left(\frac{2(n-1)\pi}{n}\right) = -1$$

and

$$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \cdots + \sin\left(\frac{2(n-1)\pi}{n}\right) = 0.$$

[*Hint: use the fact* $e^{ix} = \cos x + i \sin x$, $x \in \mathbb{R}$.]

10.*

For the mapping $w = u + iv = f(z) = 1/z$ find the images of the curves (i) $x = C$ or $y = C$, (ii) $|z| = R$, (iii) $\arg z = \alpha$, (iv) $|z - 1| = 1$.