MATH95007 Problems Sheet 2
Solutions

1a. $\frac{\sqrt{2}}{2}\,(1+i)$; $\quad 2^{1/4}e^{\pi/8}$; $\quad \frac{\sqrt{3}}{2} - i\,\frac{1}{2}$.

1b. $\sin i = (e^{-1}-e)/2i$; $\quad 2^{i} = e^{i\ln 2 - 2\pi k}$, $k \in \mathbb{Z}$; $\quad i^{i} = e^{-\pi/2+2\pi k}$, $k \in \mathbb{Z}$.

1c. $\mathrm{Log}\, i = i\pi/2$; $\quad \mathrm{Log}\,(-1-i) = \frac{1}{2}\,\ln 2 - i3\pi/4$.

1d. $\mathrm{Log}\,(z^2) \neq 2\,\mathrm{Log}\,(z)$.

2a.

$$\sin(z_1+z_2) = \frac{1}{2i}\,(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}).$$

$$\begin{aligned}
&\sin z_1 \cos z_2 + \sin z_2 + \cos z_1 \\
&\quad = \frac{1}{4i}\left((e^{iz_1} - e^{-iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{iz_1} + e^{-iz_1})(e^{iz_2} - e^{-iz_2})\right) \\
&\qquad = \frac{1}{4i}\Big(e^{iz_1+iz_2} - e^{-iz_1+iz_2} + e^{iz_1-iz_2} - e^{-iz_1-iz_2} \\
&\qquad\quad + e^{iz_1+iz_2} + e^{-iz_1+iz_2} - e^{iz_1-iz_2} - e^{-iz_1-iz_2}\Big) \\
&\qquad\qquad = \frac{1}{2i}\,(e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}).
\end{aligned}$$

2b.

$$\tan 2z = \frac{1}{i}\,\frac{e^{2iz} - e^{-2iz}}{e^{2iz} + e^{-2iz}}.$$

$$\begin{aligned}
\frac{2\tan z}{1-\tan^2 z} &= \frac{2}{i}\,\frac{\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}}{1+\left(\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}\right)^2} \\
&= \frac{2}{i}\,\frac{(e^{iz}-e^{-iz})(e^{iz}+e^{-iz})}{(e^{iz}+e^{-iz})^2 + (e^{iz}-e^{-iz})^2} = \frac{2}{i}\,\frac{e^{2iz}-e^{-2iz}}{2e^{2iz}+2e^{-2iz}}.
\end{aligned}$$

3. $\mathrm{Log}\,(-1+i/n) = \ln\sqrt{1+1/n^2} + i\theta_n$, where $\theta_n \to \pi$ and
$\mathrm{Log}\,(-1-i/n) = \ln\sqrt{1+1/n^2} + i\varphi_n$, where $\varphi_n \to -\pi$, as $n \to \infty$.

4.

(i) Note that

$$P(z) = \frac{z^n - 1}{z - 1} = 1 + z + z^2 + \cdots + z^{n-1}.$$

Therefore $P(1) = n$.

(ii) The points Q_k, $k = 0, \ldots, n-1$, can be identified with the roots of the equation $z^n = 1$ which are

$$z_k = e^{2\pi i k/n}, \quad k = 0, \ldots, n-1.$$

Consider

$$\frac{z^n - 1}{z - 1} = \frac{1}{z - 1} \prod_{k=0}^{n-1} (z - z_k) = \prod_{k=1}^{n-1} (z - z_k).$$

Clearly

$$d_k = |1 - z_k|, \quad k = 1, \ldots, n-1.$$

Therefore

$$\prod_{k=1}^{n-1} d_k = \prod_{k=1}^{n-1} |1 - z_k| = \left| \prod_{k=1}^{n-1} (1 - z_k) \right| = \left| \frac{z^n - 1}{z - 1} \right|_{z=1} = P(1) = n.$$

5a.

$$\int_{\gamma_1} z^k \, dz = \int_0^{2\pi} e^{ik\theta} \, i \, e^{i\theta} \, d\theta = \begin{cases} 2\pi i, & \text{if } k+1 = 0, \\ 0, & \text{if } k+1 \neq 0. \end{cases}$$

5b.

$$\int_{\gamma_2} z^k \, dz = \int_0^{4\pi} e^{ik\theta} \, i \, e^{i\theta} \, d\theta = \begin{cases} 4\pi i, & \text{if } k+1 = 0, \\ 0, & \text{if } k+1 \neq 0. \end{cases}$$

6a.

- $\gamma = \{z = x + iy \in \mathbb{C} : \ y = 2x, \ x \in [0, 1]\}$. Thus

$$J = \int_\gamma \operatorname{Im} z \, dz = \int_\gamma y \, d(x + iy) = \int_0^1 2x d(x + i2x)$$

$$= 2 \int_0^1 x \, dx + 4i \int_0^1 x \, dx = 1 + 2i.$$

- $\gamma = \{z = x + iy \in \mathbb{C} : y = 2x^2,\ x \in [0, 1]\}$. Then

$$J = \int_\gamma \operatorname{Im} z\, dz = \int_0^1 2x^2\, d(x + i2x^2)$$

$$= 2\int_0^1 x^2\, dx + 8i\int_0^1 x^3\, dx = \frac{2}{3} + 2i.$$

6b. We find that $\gamma = \{z = re^{i\theta} \in \mathbb{C} : r = 2,\ \theta \in [\pi/2, \pi]\}$. Then

$$J = \int_{\pi/2}^{\pi} (i \cdot 2\, e^{-i\theta} + 4\, e^{2i\theta})\, 2i\, e^{i\theta}\, d\theta = -2\pi - \frac{8}{3} + i\,\frac{8}{3}.$$

7a. The integrant $1/z$ is continuous outside $z = 0$ and, in particular, on the curve $\gamma : z = re^{i\theta}$, $-\pi < \theta \leq \pi$, and moreover $d(\operatorname{Log} z)/dz = 1/z$ outside the branch cut $(-\infty, 0]$. The curve γ does not intersect this branch cut and therefore

$$\int_\gamma \frac{1}{z}\, dz = \operatorname{Log} z\Big|_{-i}^{i} = \operatorname{Log}(i) - \operatorname{Log}(-i)$$

$$= (\ln 1 + i\,\pi/2) - ((\ln 1 - i\,\pi/2) = i\,\pi.$$

7b. Once again $1/z$ is continuous on

$$\gamma = \{z \in \mathbb{C} : z = e^{i\theta},\ \theta \in [3\pi/2, \pi/2]\}.$$

However $\operatorname{Log} z$ is not holomorphic on $(-\infty, 0]$ and in this case γ intersects this branch cut, so we *cannot* claim that $d(\operatorname{Log} z)/dz = 1/z$ at $\theta = \pi$. Let us choose another branch cut for log. Let, for example, it be $[0, +\infty)$ and denote $\log$-function with this branch cut by $\log_0 z$. Namely, in this case for $z = re^{i\theta}$ we have $0 \leq \theta < 2\pi$. Then

$$\int_\gamma \frac{1}{z}\, dz = \log_0(z)\Big|_{-i}^{i} = \log_0(i) - \log_0(-i)$$

$$= (\ln 1 + i\,\pi/2) - (\ln 1 + i\,3\pi/2) = -i\,\pi.$$

8. If $n \neq 1$, then in the domain $|z - z_0| > 0$ the integrant has primitive and

$$\int \frac{1}{(z - z_0)^n}\, dz = \frac{1}{(1 - n)(z - z_0)^{n-1}} + C.$$

Consequently , for $n \neq 1$

$$\oint_\gamma \frac{1}{(z - z_0)^n}\, dz = 0.$$

For $n = 1$, on the other hand, $1/(z - z_0)^n$ does not have a primitive in any domain that contains γ. (Every branch of $\log(z - z_0)$ has a branch cut that intersects γ). We set $z = z_0 + re^{i\theta}$, $-\pi < \theta \leq \pi$ and obtain

$$\oint_\gamma \frac{1}{z - z_0}\,dz = \int_{-\pi}^{\pi} \frac{1}{re^{i\theta}}\,i\,r\,e^{i\theta}\,d\theta = i\theta\Big|_{-\pi}^{\pi} = 2i\pi.$$

9.

$$\oint_\gamma \sqrt{z}\,dz = \int_{-\pi}^{\pi} \sqrt{3}e^{i\theta/2}\,i\,3\,e^{i\theta}\,d\theta = 3^{3/2}\,i \int_{-\pi}^{\pi} e^{i3\theta/2}\,d\theta$$

$$= 3^{3/2}\,\frac{2}{3}\,e^{i3\theta/2}\Big|_{-\pi}^{\pi} = 2\sqrt{3}(e^{i3\pi/2} - e^{-i3\pi/2}) = 2\sqrt{3}(-2\,i) = -4\sqrt{3}\,i.$$