MATH50001 - Problems Sheet 2

1. Evaluate

1a) $\sqrt{i}$, $\sqrt{1+i}$, $\sqrt{\frac{1-i\sqrt{3}}{2}}$, where $\sqrt{}$ takes its principal value.

1b) $\sin i$, 2^i, i^i, $(-1)^{2i}$, where multi-valued functions is considered.

1c) $\mathrm{Log}\, i$, $\mathrm{Log}\,(-1-i)$.

1d) Find an error in the reasoning leading to Bernoulli's paradox:
$(-z)^2 = z^2$, hence $2\,\mathrm{Log}\,(-z) = 2\,\mathrm{Log}\,(z)$ and consequently,
$\mathrm{Log}\,(-z) = \mathrm{Log}\,(z)$.

2. Prove that
2a) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$.

2b) $\tan 2z = \frac{2\tan z}{1-\tan^2 z}$.

3. Let $\mathrm{Log}\, z = \ln|z| + i\theta$, where $-\pi < \theta \le \pi$ and $z = |z|e^{i\theta}$, $(z \neq 0)$. Prove that Log is not continuous on $(-\infty, 0)$.

Hint: Consider the sequences $\{-1 + i/n\}$ and $\{-1 - i/n\}$.

4.*
(i) Let $P(z) = \frac{z^n - 1}{z-1}$. Find $P(1)$.

(ii) Let Q_k, $k = 0, \dots, n-1$, be the vertices of a regular polygon inscribed in the unit circle such that $Q_0 = 1$. Let d_k be the distance between Q_k and Q_0. Show that

$$\prod_{k=1}^{n-1} d_k = n.$$

5. Compute the integral

$$J = \int_\gamma z^k \, dz, \qquad k = 0, \pm 1, \pm 2, \dots.$$

2a) $\gamma = \gamma_1 = \{z = |z|\, e^{i\theta} \in \mathbb{C} : \, |z| = 1, \, \theta \in [0, 2\pi]\}$.

2b) $\gamma = \gamma_2 = \{z = |z|\, e^{i\theta} \in \mathbb{C} : \, |z| = 1, \, \theta \in [0, 4\pi]\}$.

6.

6a) Compute the integral $J = \int_\gamma \mathrm{Im}\, z \, dz$, where
- γ is the an interval between $w_1 = 0$ and $w_2 = 1 + 2i$.
- γ is a part of the parabola $y = 2x^2$ connecting $w_1 = 0$ and $w_2 = 1 + 2i$.

6b. Compute the integral

$$J = \int_\gamma (i\bar{z} + z^2)\, dz,$$

where γ is a part of the circle $|z| = 2$, $\arg z \in [\pi/2, \pi]$.

7.* Evaluate the contour integrals:

$$\int_\gamma \frac{1}{z}\, dz,$$

where
  a) $\gamma = \{z \in \mathbb{C} : z = e^{i\theta},\ \theta \in [-\pi/2, \pi/2]\}$,
  b) $\gamma = \{z \in \mathbb{C} : z = e^{i\theta},\ \theta \in [3\pi/2, \pi/2]\}$.

Integrals over closed curves below are with counterclockwise orientation:
8.

$$\oint_\gamma \frac{1}{(z - z_0)^n}\, dz, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

where $\gamma = \{z \in \mathbb{C} : |z - z_0| = r\}$, $r > 0$.

9.

$$\oint_\gamma \sqrt{z}\, dz,$$

where $\gamma = \{z \in \mathbb{C} : |z| = 3\}$.