MATH50001 Problems Sheet 3
Solutions

1). Because $1/(z^2-4)$ is holomorphic at all points

$$\{z:\ |z|\le 4\}\setminus \Big\{\{z:\ |z+2|\le 1\}\cup\{z:\ |z-2|\le 1\}\Big\}.$$

we obtain

$$\oint_\gamma \frac{1}{z^2-4}\,dz=\oint_{\gamma_1}\frac{1}{(z+2)(z-2)}\,dz+\oint_{\gamma_2}\frac{1}{(z+2)(z-2)}\,dz,$$

where $\gamma_1:\ |z+2|=1$ and $\gamma_2:\ |z-2|=1$.

Note that

$$\frac{1}{(z+2)(z-2)}=-\frac{1}{4}\frac{1}{z+2}+\frac{1}{4}\frac{1}{z-2}.$$

It follows that

$$\oint_\gamma \frac{1}{z^2-4}\,dz=\frac{1}{4}\oint_{\gamma_1}\left(\frac{1}{z-2}-\frac{1}{z+2}\right)dz+\frac{1}{4}\oint_{\gamma_2}\left(\frac{1}{z-2}-\frac{1}{z+2}\right)dz.$$

Because $1/(z-2)$ is holomorphic inside and on γ_1 its integral around γ_1 is zero (by the Cauchy-Goursat theorem). Similarly, the contour integral of $1/(z+2)$ around γ_2 vanishes. Besides,

$$\oint_{\gamma_1}\frac{1}{z+2}\,dz=2\pi i \quad\text{and}\quad \oint_{\gamma_2}\frac{1}{z-2}\,dz=2\pi i.$$

Consequently

$$\oint_\gamma \frac{1}{z^2-4}\,dz=-\frac{1}{4}2\pi i+\frac{1}{4}2\pi i=0.$$

2).

$$\oint_\gamma \frac{1}{z^3-1}\,dz=\oint_\gamma \frac{1}{(z-1)(z-e^{i2\pi/3})(z-e^{i4\pi/3})}\,dz.$$

Note that

$$\frac{1}{(z-1)(z-e^{i2\pi/3})(z-e^{i4\pi/3})}$$
$$=\frac{1}{3}\frac{1}{z-1}-\frac{e^{-i\pi/3}}{3}\frac{1}{z-e^{i2\pi/3}}+\frac{e^{i\pi/3}}{3}\frac{1}{z-e^{i4\pi/3}}$$

Both functions $1/(z-1)$ and $1/(z-e^{i4\pi/3})$ are holomorphic inside and on $\gamma=\{z:\ |z-i|=1\}$. Therefore

$$\oint_\gamma \frac{1}{z-1}\,dz=\oint_\gamma \frac{1}{z-e^{i4\pi/3}}\,dz=0.$$

Since $e^{i2\pi/3}$ is inside the domain bounder by γ we have

$$\oint_\gamma \frac{1}{z-e^{i2\pi/3}}\,dz = 2\pi i.$$

Thus

$$\oint_\gamma \frac{1}{z^3-1}\,dz = -\frac{e^{-i\pi/3}}{3}\,2\pi i = -\frac{1-i\sqrt{3}}{6}\,2\pi i = -\frac{\pi(\sqrt{3}+i)}{3}.$$

3). a) No, b) Yes, c) No.

4).

$$\oint_\gamma \frac{e^z \sin z}{z-5}\,dz = 2\pi i e^z \sin z\Big|_{z=5} = 2\pi i e^5 \sin 5.$$

5). Let $\operatorname{Re} z > 0$, choose any $\beta > |z|$. We consider the interval $\gamma_1 = [i\beta, -i\beta]$, half circle

$$\gamma_2 = \{z:\ z = \beta\, e^{i\theta},\ -\pi/2 < \theta < \pi/2\}$$

and denote $\gamma = \gamma_1 \cup \gamma_2$. Then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)}{\eta - z}\,d\eta = \frac{1}{2\pi i}\int_{i\beta}^{-i\beta} \frac{f(\eta)}{\eta - z}\,d\eta + \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\eta)}{\eta - z}\,d\eta.$$

Let $r = \min_{\eta\in\gamma_2} |\eta - z|$. Using our assumptions we find that if $\eta \in \gamma_2$

$$\left|\frac{f(\eta)}{\eta - z}\right| \le \frac{M}{|\eta|^k}\,\frac{1}{|\eta - z|} \le \frac{M}{\beta^k\, r}.$$

According to the ML-inequality

$$\left|\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\eta)}{\eta - z}\,d\eta\right| \le \frac{M}{2\pi\beta^k\, r}\cdot \pi\beta = \frac{M}{2\beta^{k-1}\, r} \to 0,$$

as $\beta \to \infty$. Thus

$$f(z) = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{i\beta}^{-i\beta} \frac{f(\eta)}{\eta - z}\,d\eta = -\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{-i\beta}^{i\beta} \frac{f(\eta)}{\eta - z}\,d\eta.$$

6). We use Cauchy's theorem

$$f(z_0) = \frac{1}{2i\pi}\oint_{|z|=1} \frac{f(z)}{z-z_0}\,dz$$
$$= \frac{1}{2i\pi}\oint_{|z|=1} \frac{f(z)}{z-z_0}\,dz - \frac{1}{2i\pi}\oint_{|z|=1} \frac{f(z)}{z-\bar{z}_0^{-1}}\,dz$$
$$= \frac{1}{2i\pi}\oint_{|z|=1} \frac{-\bar{z}_0^{-1}+z_0}{(z-z_0)(z-\bar{z}_0^{-1})} f(z)\,dz$$
$$= \frac{1}{2i\pi}\oint_{|z|=1} \frac{-\bar{z}_0^{-1}(1-|z_0|^2)}{(z-z_0)(z-\bar{z}_0^{-1})} f(z)\,dz$$
$$= \frac{1}{2i\pi}\oint_{|z|=1} \frac{(1-|z_0|^2)}{(z-z_0)(1-z\bar{z}_0)} f(z)\,dz.$$

7)

Let

$$\gamma_1 = \{z : z = 2e^{i\varphi}, \varphi \in [0,\pi]\}, \qquad \gamma_2 = \{z : z = x,\ -2 < x < -1\},$$
$$\gamma_3 = \{z : z = e^{i\varphi}, \varphi \in [\pi, 0]\}, \qquad \gamma_4 = \{z : z = x,\ 1 < x < 2\}.$$

Then

$$\int_{\gamma_1} \frac{z}{\bar{z}}\,dz = \int_0^\pi \frac{2e^{i\varphi}}{2e^{-i\varphi}}\, 2i\, e^{i\varphi}\,d\varphi = 2i\int_0^\pi e^{3i\varphi}\,d\varphi = \frac{2}{3}\left(e^{3i\pi}-1\right) = -\frac{4}{3},$$
$$\int_{\gamma_2} \frac{z}{\bar{z}}\,dz = \int_{-2}^{-1} dx = x\Big|_{-2}^{-1} = -1-(-2) = 1,$$
$$\int_{\gamma_3} \frac{z}{\bar{z}}\,dz = \int_\pi^0 \frac{e^{i\varphi}}{e^{-i\varphi}}\, i\, e^{i\varphi}\,d\varphi = i\int_\pi^0 e^{3i\varphi}\,d\varphi = \frac{1}{3}\left(1-e^{3i\pi}\right) = \frac{2}{3},$$
$$\int_{\gamma_4} \frac{z}{\bar{z}}\,dz = \int_1^2 dx = x\Big|_1^2 = 1.$$

Finally we have

$$\int_\gamma \frac{z}{\bar{z}}\,dz = \sum_{j=1}^{4}\int_{\gamma_i} \frac{z}{\bar{z}}\,dz = -\frac{4}{3}+1+\frac{2}{3}+1 = \frac{4}{3}.$$

8a)

Let us introduce curves $\gamma_n = \{z : |z-n| = 1/2\}$, $n = 0, 1, \dots k$. Then by using the Deformation Theorem we have

$$I_k = \oint_\gamma \frac{dz}{z(z-1)\dots(z-k)}\, dz = \sum_{n=0}^{k} \oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)}\, dz.$$

Note that for any $n : 0 \le n \le k$ the Cauchy's integral formula implies

$$\oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)}\, dz = \frac{1}{n} \dots \frac{1}{1} \cdot \frac{1}{-1} \dots \frac{1}{n-k} = 2\pi i \frac{(-1)^{k-n}}{n!(k-n)!}.$$

Thus

$$\sum_{n=0}^{k} \oint_{\gamma_n} \frac{dz}{z(z-1)\dots(z-k)}\, dz = 2\pi i \sum_{n=0}^{k} \frac{(-1)^{k-n}}{n!(k-n)!}.$$

8b)

Using Cauchy's integral formula we obtain (for $k \ge 1$)

$$J_k = \oint_\gamma \frac{(z-1)(z-2)\dots(z-k)}{z} = 2\pi i\, (-1)^k\, k!.$$

9. *Step 1.* We first prove that if $\gamma \subset \mathbb{C}$ is a simple closed piecewise-smooth curve bounding Ω and f and $f'_{\bar z}$ are continuous inside and on γ then

$$\oint_\gamma f(z)\, dz = 2i \iint_\Omega \frac{\partial f(z)}{\partial \bar z}\, dx dy.$$

Indeed, this is a direct corollary of Green's theorem

$$\oint_\gamma f(z)\, dz = \oint_\gamma (u + iv)\, d(x+iy) = \oint_\gamma u\, dx - v\, dy + i \oint_\gamma v\, dx + u\, dy$$

$$= \iint_\Omega (-v'_x - u'_y + i\, u'_x - i\, v'_y)\, dx dy$$

$$= 2i \iint_\Omega \frac{1}{2}\left(u'_x - \frac{1}{i} u'_y\right) + \frac{i}{2}\left(v'_x - \frac{1}{i} v'_y\right) dx dy$$

$$= 2i \iint_\Omega \frac{d}{d\bar z}\left(u + iv\right) dx dy = 2i \iint_\Omega \frac{d}{d\,\bar z} f(z)\, dx dy.$$

Step 2. Let $r > 0$ is small enough, so that $B_r(z_0) = \{z : |z - z_0| \le r\} \subset D$. Using the same argument as in the proof of the Deformation theorem we find

$$\oint_{|z-z_0|=1} \frac{f(z)}{z-z_0}\,dz - \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0}\,dz$$
$$= 2i \iint_{D\setminus B_r(z_0)} \frac{d}{d\bar{z}}\left(\frac{f(z)}{z-z_0}\right) dx\,dy = 2i \iint_{D\setminus B_r(z_0)} \frac{df(z)/d\bar{z}}{z-z_0}\,dx\,dy,$$

where we have used the fact that $1/(z-z_0)$ is holomorphic in $D \setminus B_r(z_0)$ and therefore $d(z-z_0)^{-1}/d\bar{z} = 0$. It remains to note that

$$\lim_{r\to 0} \oint_{|z-z_0|=r} \frac{f(z)}{z-z_0}\,dz = \lim_{r\to 0} \oint_{|z-z_0|=r} \frac{f(z_0)+f(z)-f(z_0)}{z-z_0}\,dz = 2\pi i f(z_0)$$

and that

$$\lim_{r\to 0} \iint_{D\setminus B_r(z_0)} \frac{df(z)/d\bar{z}}{z-z_0}\,dx\,dy = \iint_{D} \frac{df(z)/d\bar{z}}{z-z_0}\,dx\,dy.$$