MATH50001 - Problems Sheet 3

1. Find
$$\oint_\gamma \frac{1}{z^2-4}\,dz,$$
where $\gamma = \{z \in \mathbb{C} : |z| = 4\}$.

2. Compute
$$\oint_\gamma \frac{1}{z^3-1}\,dz,$$
where $\gamma = \{z \in \mathbb{C} : |z-i| = 1\}$.

3. Determine whether the domain is simply connected:

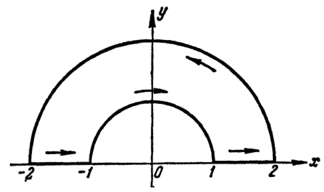
a) $1 < |z-3| < 2$, b) $|z+1| + |z-1| < 4$, c) $|z| > 5$.

4. Evaluate:
$$\oint_\gamma \frac{e^z \sin z}{z-5}\,dz,$$
where $\gamma = \{z \in \mathbb{C} : |z| = 6\}$.

5. Suppose that f is holomorphic in the half-plane $\{z : \operatorname{Re} z \geq 0\}$ and that in this half-plane, there exist M, R and $k > 1$ such that
$$|f(z)\, z^k| < M, \quad \text{for} \quad |z| > R.$$
Prove that if $\operatorname{Re} z > 0$, then
$$f(z) = -\frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{-i\beta}^{i\beta} \frac{f(\eta)}{\eta - z}\,d\eta.$$

6.* Let f be a holomorphic function on a neighborhood of the disc $\mathbb{D} = \{z : |z| \leq 1\}$. Prove the integral formula
$$f(z_0) = \frac{1}{2i\pi} \oint_{|z|=1} \frac{1-|z_0|^2}{(z-z_0)(1-\bar{z}_0 z)}\, f(z)\,dz, \qquad |z_0| < 1.$$

7. Evaluate the integral
$$\oint_\gamma \frac{z}{\bar{z}}\,dz,$$
where γ is the boundary of the half ring



8.

Let γ be a simple closed contour enclosing the points $0, 1, 2, \ldots, k$ in the complex plane. Evaluate the integrals

a)

$$I_k = \oint_\gamma \frac{dz}{z(z-1)\ldots(z-k)}, \qquad k = 0, 1, \ldots.$$

b)

$$J_k = \oint_\gamma \frac{(z-1)(z-2)\ldots(z-k)}{z}\, dz, \qquad k = (0)1, 2\ldots.$$

9.* Let Ω be an open set such that $D = \{z : |z - z_0| \leq 1\} \subset \Omega$. Show that if f and $\partial f/\partial \bar{z}$ are continuous in Ω and $\gamma = \{z : |z - z_0| = 1\}$, then

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz - \frac{1}{\pi} \iint_D \frac{df(z)/d\bar{z}}{z - z_0}\, dx dy.$$