MATH50001 - Problems Sheet 4

1. Evaluate

$$\oint_{\gamma} \frac{z^2}{(z-1)^n}\, dz, \qquad n = 1, 2, \ldots,$$

where $\gamma = \{z : |z| = 2\}$.

2.

2a) Find the curve $\gamma = \{z \in \mathbb{C} : |z-2| + |z+2| = 10\}$.

2b) Evaluate the contour integral:

$$\oint_{\gamma} \frac{\sin z}{(z+2)^3}.$$

3.* Show that for every polynomial $p(z)$,

$$\max_{\{z:|z|=1\}} |z^{-1} - p(z)| \geq 1.$$

4. Let $f(z)$ be a bounded entire function. Compute for R large and $z_0, z_1 \in \mathbb{C}$ ($|z_0|, |z_1| < R$) the value of the integral

$$\frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{(z-z_0)(z-z_1)}\, dz.$$

By taking $R \to \infty$, obtain a new proof of Liouville's theorem.

5. Let f be an entire function such that $|f(z)| \leq C(1+|z|)^n$, for some $n \in \mathbb{N}$ and where $C \geq 0$. Show that f is a polynomial of degree smaller or equal than n.

6. Prove that if an entire function has a bounded real part or a bounded imaginary part, then it is constant.

7. Determine whether the series converges or diverges:

a. $\sum_{n=1}^{\infty} \frac{1}{n^2+i}$.

b. $\sum_{n=1}^{\infty} \frac{3-(2i)^n}{\cos ni}$.

c. $\sum_{n=1}^{\infty} \left(\frac{ni}{n+i}\right)^{n^2}$.

8. For what values of z does the series $\sum_{n=1}^{\infty} \frac{e^{nz}}{(n+1)^2}$ converge?

9. Find the circle of convergence for the power series:

a. $\sum_{n=1}^{\infty} \frac{1}{(n+i)^3} z^n$

b. $\sum_{n=1}^{\infty} (1+i)^n (z-4)^{2n}$

c. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+i}} (z-2)^n$.

10. Find the power series of the function

a. $f(z) = \frac{z^2}{2-z}$ about $z_0 = 0$.

b. $f(z) = \frac{1}{1+z}$ about $z_0 = i$.

11. Find the Taylor series for:

a. $\cos z$ about $z_0 = 0$.

b. e^z about $z_0 = 1 + i$.

c. $\operatorname{Log} z$ about $z_0 = i$.

12.* Let f be holomorphic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let its Taylor series at 0 is

$$f(z) = \sum_{k=0}^{\infty} a_n z^n.$$

a. Prove that $|f(z)| < 1$ implies $|a_k| \leq 1$.

b. Let now

$$|f(z)| < \frac{1}{1 - |z|}.$$

What is the best bound you can get from Cauchy estimates for $|a_k|$? How does the bound behave as $n \to \infty$?