MATH50001 Problems Sheet 5
Solutions

1)
a)

$$\frac{1}{2} \cdot \frac{1}{z-2i} + \sum_{n=0}^{\infty} \frac{i^{n-1}}{2^{2n+3}} (z-2i)^n, \qquad 0 < |z-2i| < 4.$$

b)

$$\sum_{n=-1}^{\infty} \frac{1}{e(n+1)!} (z+1)^n, \qquad |z+1| > 0.$$

2)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n (z-1)^n} - \sum_{n=0}^{\infty} (z-1)^n.$$

3) Obviously

$$\frac{9}{(z-4)(z+5)} = \frac{1}{z-4} - \frac{1}{z+5}.$$

3a) If $|z| < 4$ we have

$$\frac{1}{z-4} = -\frac{1}{4-z} = -\frac{1}{4} \frac{1}{1-z/4} = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{4^n} z^n$$

and

$$-\frac{1}{z+5} = -\frac{1}{5} \frac{1}{1-(-z/5)} = -\frac{1}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} z^n.$$

Therefore

$$\frac{9}{(z-4)(z+5)} = -\sum_{n=0}^{\infty} \left(\frac{1}{4^{n+1}} + \frac{(-1)^n}{5^{n+1}} \right) z^n.$$

3b) If $2 < |z| < 5$, then

$$\frac{1}{z-4} = \frac{1}{z} \frac{1}{1-4/z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{4^n}{z^n}$$

and thus

$$\frac{9}{(z-4)(z+5)} = \sum_{n=0}^{\infty} \frac{4^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} z^n$$

$$= \sum_{n=-\infty}^{-1} 4^{-n-1} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} z^n.$$

3c) If $5 < |z|$, then

$$-\frac{1}{z+5} = -\frac{1}{z}\frac{1}{1-(-5/z)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{(-5)}{z}\right)^n$$

which implies

$$\frac{9}{(z-4)(z+5)} = \sum_{n=0}^{\infty} \frac{4^n}{z^{n+1}} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{(-5)}{z}\right)^n$$

$$= \sum_{n=-\infty}^{-1} \left(4^{-n-1} - (-5)^{-n-1}\right) z^n.$$

4) Let $f(z) = z\,e^z$ and $z_0 = 2$. Then

$$f(z) = (z-2)\,e^{z-2}e^2 + 2e^{z-2}e^2 = \sum_{n=0}^{\infty} \frac{e^2}{n!}(z-2)^{n+1} + \sum_{n=0}^{\infty} \frac{2e^2}{n!}(z-2)^n$$

$$= 2e^2 + \sum_{n=1}^{\infty}\left(\frac{e^2}{(n-1)!} + \frac{2e^2}{n!}\right)(z-2)^n.$$

5)

a) If f is holomorphic at z_0 and has a zero of order m at z_0, then there is $g(z)$ holomorphic at z_0, $g(z_0) \neq 0$ such that $f(z) = (z-z_0)^m\,g(z)$. Therefore

$$\frac{1}{f(z)} = \frac{1}{(z-z_0)^m}\cdot\frac{1}{g(z)}.$$

Thus $1/f$ has a pole of order m at z_0.

b)

$$\left(2\cos z - 2 - z^2\right)^2 = \left(2\left(1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \dots\right) - 2 + z^2\right)^2$$

$$= 4\left(\frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \dots\right)^2 = z^8\cdot g(z),$$

where $g(z)$ is holomorphic at 0 and $g(0) \neq 0$. Therefore $\frac{1}{(2\cos z - 2 + z^2)^2}$ has a pole at 0 of order 8.

6)

a) $z = 0$, essential singularity;

b) $z = 0$, pole of order 4;

c) $z = n\pi$, $(6n \pm 1)\pi/3$, poles of order 1.

7) The function $f(z) = \frac{e^z}{z(z-2)^3}$ has two poles inside $\gamma = \{|z| = 3\}$. One of them is at $z_1 = 0$ of order one and the other one is at $z_2 = 2$ of order three. Therefore

$$\begin{aligned}\oint_\gamma \frac{e^z}{z(z-2)^3}\,dz &= 2\pi i\,(\mathrm{Res}\,[f, z_1] + \mathrm{Res}\,[f, z_2]) \\ &= 2\pi i \left(\frac{e^0}{(-2)^3} + \lim_{z\to 2} \frac{1}{2} \frac{d^2}{dz^2} \frac{(z-2)^3 e^z}{z(z-2)^3} \right) \\ &= -\frac{\pi i}{4} + \pi i \lim_{z\to 2} \left(\frac{e^z}{z} - 2\frac{e^z}{z^2} + 2\frac{e^z}{z^3} \right) \\ &= -\frac{\pi i}{4} + \pi i\, e^2 \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4} \right) = \frac{\pi(e^2-1)i}{4}.\end{aligned}$$

8) Substituting $z = e^{i\theta}$ gives

$$\begin{aligned} I := \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} &= \frac{1}{i} \oint_{|z|=1} \frac{1}{-az^2 + (1+a^2)z - a}\,dz \\ &= \frac{1}{i} \oint_{|z|=1} \frac{1}{(z-a)(1-az)}\,dz.\end{aligned}$$

If $|a| < 1$ there is a pole inside $|z| = 1$ at a with the residue $1/(1-a^2)$. Therefore $I = 2\pi/(1-a^2)$.
If $|a| > 1$ the pole inside $|z| = 1$ is at $1/a$ and the residue is

$$\lim_{z\to 1/a} \frac{z - 1/a}{(z-a)(az-1)} = \frac{1}{a^2-1}.$$

Hence $I = 2\pi/(a^2-1)$.

9)

$$\oint_\gamma \frac{e^z-1}{z^2(z-1)} = 2\pi\,i\,2\,\mathrm{Res}\left[\frac{e^z-1}{z^2(z-1)},1\right] - 2\pi\,i\,2\,\mathrm{Res}\left[\frac{e^z-1}{z^2(z-1)},0\right]$$
$$= 4\pi i\left\{\frac{e^z-1}{z^2}\Big|_{z=1} - \frac{z}{dz}\frac{e^z-1}{z-1}\Big|_{z=0}\right\}$$
$$= 4\pi i\left\{e-1-\frac{e^z(z-1)-(e^z-1)}{(z-1)^2}\Big|_{z=0}\right\}$$
$$= 4\pi i\left\{e-1+1\Big|_{z=0}\right\} = 4\pi i e.$$

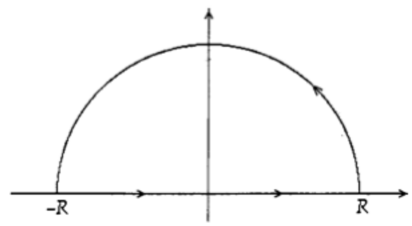
10) Indeed,

$$\frac{1}{2\pi i}\oint_{|z|=r} z^{n-1}\,|f(z)|^2\,dz = \frac{1}{2\pi i}\int_0^{2\pi} r^{n-1}\,e^{i(n-1)\theta}\,|f(re^{i\theta})|^2\,i\,e^{i\theta}\,r\,d\theta$$
$$= \frac{r^n}{2\pi}\int_0^{2\pi} e^{i(n-1)\theta}\left(\sum_{k=0}^{n} a_k r^k e^{ik\theta}\right)\overline{\left(\sum_{m=0}^{n} a_m r^m e^{im\theta}\right)}\,e^{i\theta}\,d\theta.$$

The only term that survive if $n-1+k-m+1=0$. The only possibility for that is $k=0$ and $m=n$ and thus

$$\frac{1}{2\pi i}\oint_{|z|=r} z^{n-1}\,|f(z)|^2\,dz = a_0\,\bar{a}_n\,r^{2n}.$$

11)
a. Let first $\xi < 0$ and consider
$$\oint_{\gamma} \frac{e^{-i\xi z}}{1+z^2}\, dz,$$
where $\gamma = \gamma_1 \cup \gamma_2$

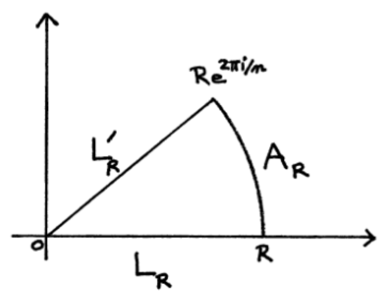


$$\gamma_1 = \{z:\ z = x + i0,\ -R < x < R\},$$
$$\text{and} \quad \gamma_2 = \{z:\ z = R\, e^{i\theta},\ 0 \le \theta \le \pi\}, \quad R > 1.$$
Then
$$\oint_{\gamma} \frac{e^{-i\xi z}}{1+z^2}\, dz = 2\pi i \,\mathrm{Res}\left[\frac{e^{-i\xi z}}{1+z^2}, i\right] = 2\pi i\, \frac{e^{\xi}}{2i} = \pi\, e^{-|\xi|}.$$
Note that since $0 \le \theta \le \pi$ we have $\sin\theta > 0$. Therefore by using the ML-inequality we find
$$\left|\int_{\gamma_2} \frac{e^{-i\xi z}}{1+z^2}\, dz\right| \le \pi R \max\left|\frac{e^{-i\xi R(\cos\theta + i\sin\theta)}}{1+R^2 e^{2i\theta}}\right| \le \frac{\pi R}{R^2-1} \to 0,$$
as $R \to \infty$.
Finally
$$\pi\, e^{-|\xi|} = \pi\, e^{\xi} = \oint_{\gamma} \frac{e^{-i\xi z}}{1+z^2}\, dz = \lim_{R\to\infty}\left(\int_{\gamma_1} \frac{e^{-i\xi z}}{1+z^2}\, dz + \int_{\gamma_2} \frac{e^{-i\xi z}}{1+z^2}\, dz\right)$$
$$= \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{e^{-i\xi x}}{1+x^2}\, dx + \int_{\gamma_2} \frac{e^{-i\xi z}}{1+z^2}\, dz\right) = \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2}\, dx.$$

b. If $\xi > 0$ then by substituting $x = -y$ we have
$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2}\, dx = \int_{-\infty}^{\infty} \frac{e^{-i(-\xi)y}}{1+y^2}\, dy.$$
and thus reduce the the problem to the case **1.a**.

12) Consider
$$\oint_{\gamma} \frac{1}{1+z^n}\, dz,$$

where $\gamma = \gamma_1 \cup \gamma_2$ is defined by



$$\gamma_1 = \{z : z = x + i0,\ 0 < x < R\}, \quad R > 1,$$
$$\gamma_2 = \{z : z = R\, e^{i\theta},\ 0 \le \theta \le 2\pi/n\},$$
$$\gamma_3 = \{z : z = r\, e^{i2\pi/n},\ r \in [R, 0]\}.$$

The only singularity of the function $1/(1+z^n)$ internal for γ is the point $e^{i\pi/n}$. Therefore

$$\oint_\gamma \frac{1}{1+z^n}\, dz = 2\pi i \operatorname{Res}\left[\frac{1}{1+z^n}, e^{i\pi/n}\right] = 2\pi i \frac{1}{n e^{i\pi(n-1)/n}} = -\frac{2\pi i}{n} e^{\pi i/n}.$$

Moreover

$$\int_{\gamma_1} \frac{1}{1+z^n}\, dz \to \int_0^\infty \frac{1}{1+x^n}\, dx, \qquad R \to \infty,$$
$$\int_{\gamma_2} \frac{1}{1+z^n}\, dz \to 0, \qquad R \to \infty$$

and

$$\int_{\gamma_3} \frac{1}{1+z^n}\, dz \to -e^{2\pi i/n} \int_0^\infty \frac{1}{1+x^n}\, dx, \qquad R \to \infty.$$

Finally we obtain

$$(1 - e^{2\pi i/n}) \int_0^\infty \frac{1}{1+x^n}\, dx = -\frac{2\pi i}{n} e^{\pi i/n},$$

which is equivalent to

$$\frac{e^{\pi i/n} - e^{-\pi i/n}}{2i} \int_0^\infty \frac{1}{1+x^n}\, dx = \frac{\pi}{n}.$$