MATH50001 - Problems Sheet 5

1. Find the Laurent series about the given point

a. $z/(z^2+4)$ about $z = 2i$,

b. $e^z/(z+1)$ about $z = -1$.

2. Find the Laurent series for

$$f(z) = \frac{3z-3}{2z^2-5z+2}$$

convergent for $1/2 < |z-1| < 1$.

3. Let $f(z) = \frac{9}{(z-4)(z+5)}$. Find the Laurent series for f:

(a) in the disc $|z| < 4$.

(b) in the annulus $4 < |z| < 5$.

(c) in the region $5 < |z|$.

4. Find the Taylor expansion for $f(z) = z\,e^z$ at $z_0 = 2$.

5.

a. Prove that if f is holomorphic at z_0 and has a zero of order m at z_0, then $1/f$ has a pole of order m at z_0.

b. Determine the order of the pole at $z = 0$ for

$$\frac{1}{(2\cos z - 2 + z^2)^2}.$$

6. Find and classify all isolated singularities of the functions

a. $z^3\,e^{1/z}$.

b. $\sin 3z/z^5$.

c. $1/(\sin z - \sin 2z)$.

7. Use residue theorem to evaluate

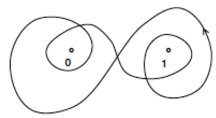
$$\oint_\gamma \frac{e^z}{z(z-2)^3}\,dz, \qquad \gamma = \{|z| = 3\}.$$

8. Evaluate

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2},$$

(i) if $|a| < 1$; (ii) if $|a| > 1$.

9.* Evaluate $\oint_\gamma \frac{e^z-1}{z^2(z-1)}$, where γ is the closed curve shown below



10.* Let f be a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$. Prove that

$$\frac{1}{2\pi i} \oint_{|z|=r} z^{n-1} \, |f(z)|^2 \, dz = a_0 \bar{a}_n \, r^{2n}.$$

11. Prove

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2} \, dx = \pi \, e^{-|\xi|}, \qquad \xi \in \mathbb{R}.$$

Show also that the ‘inverse Fourier transform” of $\pi e^{-|\xi|}$ equals

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \, e^{-|\xi|} \, e^{ix\xi} \, d\xi = \frac{1}{1+x^2}.$$

12.* Find that for any $n = 2, 3, 4, \ldots$ we have

$$\int_0^{\infty} \frac{1}{1+x^n} \, dx = \frac{\pi/n}{\sin \pi/n}.$$