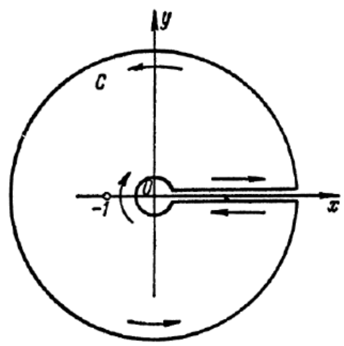
MATH50001 Problems Sheet 6
Solutions

1) Consider

$$f(z) = \frac{z^{a-1}}{1+z}, \quad z = re^{i\theta}, \quad 0 \le \theta < 2\pi$$

and consider the contour



$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4,$$

where

$$\gamma_1 = \{z : z = re^{i0}, \, r \in [\varepsilon, R]\}, \quad R > 1,$$
$$\gamma_2 = \{z : z = R\, e^{i\theta}, \, 0 \le \theta < 2\pi\},$$
$$\gamma_3 = \{z : z = r\, e^{i2\pi}, \, r \in [R, \varepsilon)]\},$$
$$\gamma_4 = \{z : z = \varepsilon\, e^{i\theta}, \, \theta \in (2\pi, 0]\}.$$

Then

$$\oint_\gamma \frac{z^{a-1}}{1+z}\, dz = 2\pi i \operatorname{Res}\left[\frac{z^{a-1}}{1+z}, e^{i\pi}\right] = 2\pi i e^{i\pi(a-1)} = -2\pi i\, e^{i\pi a}.$$

Moreover, as $\varepsilon \to 0$ and $R \to \infty$ we obtain

$$\int_{\gamma_1} \frac{z^{a-1}}{1+z}\, dz = \int_\varepsilon^R \frac{r^{a-1}}{1+r}\, dr \to \int_0^\infty \frac{r^{a-1}}{1+r}\, dr,$$

$$\Big| \int_{\gamma_2} \frac{z^{a-1}}{1+z}\, dz \Big| \le 2\pi R \frac{R^{a-1}}{R-1} \to 0,$$

$$\int_{\gamma_3} \frac{z^{a-1}}{1+z}\, dz = \int_R^\varepsilon \frac{r^{a-1}\, e^{i2\pi(a-1)}}{1+r\, e^{i2\pi}}\, e^{i2\pi}\, dr \to -e^{i2\pi a} \int_0^\infty \frac{r^{a-1}}{1+r}\, dr,$$

$$\Big| \int_{\gamma_4} \frac{z^{a-1}}{1+z}\, dz \Big| \le 2\pi \varepsilon \frac{\varepsilon^{a-1}}{1-\varepsilon} \to 0.$$

Therefore we have

$$\left(1 - e^{i2\pi a}\right) \int_0^\infty \frac{x^{a-1}}{1+x}\, dx = -2\pi\, i\, e^{i\pi a}.$$

and finally

$$\sin \pi a \int_0^\infty \frac{x^{a-1}}{1+x}\, dx = \pi.$$

2) Let

$$\gamma_1 = \{z = x + iy : \, -R \le x \le R,\; y = 0\},$$
$$\gamma_2 = \{z : \; z = R\, e^{i\theta},\; 0 < \theta < \pi\}, \qquad R > 1,$$

and $\gamma = \gamma_1 \cup \gamma_2$. Let

$$f(z) = \frac{z-1}{z^5 - 1}.$$

The simple poles of f in the upper half-plane are at the points $z_1 = e^{2i\pi/5}$ and $z_2 = e^{4i\pi/5}$. The point $z = 1$ is a removable singularity of f. Therefore

$$\oint_\gamma \frac{z-1}{z^5-1}\, dz = 2i\pi \Big(\text{Res}\,[f, z_1] + \text{Res}\,[f, z_2] \Big)$$
$$= 2i\pi \Big(\lim_{z \to z_1} f(z)(z - z_1) + \lim_{z \to z_2} f(z)(z - z_2) \Big)$$

By using l'Hopital's rule we find

$$\lim_{z \to z_1} f(z)(z - z_1) = \frac{[(z-1)(z-z_1)]'}{[(z^5-1)]'}\Big|_{z=z_1} = \frac{e^{2i\pi/5} - 1}{5\,(e^{2i\pi/5})^4}$$
$$= \frac{e^{2i\pi/5}(e^{2i\pi/5} - 1)}{5\, e^{2i\pi}} = \frac{e^{4i\pi/5} - e^{2i\pi/5}}{5}$$

and also

$$\lim_{z \to z_2} f(z)(z - z_2) = \frac{e^{4i\pi/5} - 1}{5(e^{4i\pi/5})^4} = \frac{e^{8i\pi/5} - e^{4i\pi/5}}{5}.$$

Therefore

$$\oint_\gamma \frac{z-1}{z^5-1}\, dz = \frac{2i\pi}{5} \Big(e^{4i\pi/5} - e^{2i\pi/5} + e^{8i\pi/5} - e^{4i\pi/5} \Big)$$
$$= \frac{2i\pi}{5} \Big(- e^{2i\pi/5} + e^{-2i\pi/5} \Big)$$
$$= -\frac{2i\pi}{5}\, 2i\, \sin(2\pi/5) = \frac{4\pi}{5}\, \sin(2\pi/5).$$

Moreover, if $z \in \gamma_2$, then by using the ML inequality we obtain

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \int_{\gamma_2} |f(z)|\,dz = \int_{\gamma_2} \left|\frac{z-1}{z^5-1}\right| \leq \pi R \max_{z\in\gamma_2} \left|\frac{z-1}{z^5-1}\right|$$

$$\leq \pi R \frac{R+1}{R^5-1} \to 0, \quad \text{as} \quad R \to \infty.$$

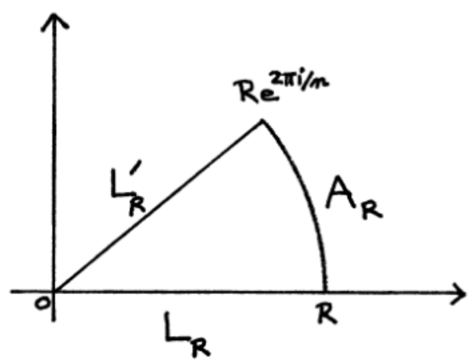
Thus

$$\int_{-\infty}^{\infty} \frac{x-1}{x^5-1}\,dx = \lim_{R\to\infty} \left(\oint_{\gamma} \frac{z-1}{z^5-1}\,dz - \int_{\gamma_2} f(z)\,dz \right) = \frac{4\pi}{5} \sin(2\pi/5).$$

3) Consider

$$\oint_{\gamma} e^{iz^2}\,dz,$$

where $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$,



$$\gamma_1 = \{z:\ z = x + i0,\ 0 < x < R\},$$

$$\gamma_2 = \{z:\ z = Re^{i\theta},\ 0 \leq \theta \leq \pi/4\},$$

$$\gamma_3 = \{z:\ z = te^{i\pi/4},\ t \in (R, 0]\}.$$

Since e^{iz^2} is holomorphic we obtain

$$\oint_{\gamma} e^{iz^2}\,dz = \int_{\gamma_1} e^{iz^2}\,dz + \int_{\gamma_2} e^{iz^2}\,dz + \int_{\gamma_3} e^{iz^2}\,dz =: I_1 + I_2 - I_3 = 0.$$

Note that

$$I_1 = \int_0^R e^{ix^2}\,dx \quad \text{and} \quad I_3 = \int_0^R e^{i(e^{i\pi/4}t)^2}\, e^{i\pi/4}\,dt = \frac{1+i}{\sqrt{2}} \int_0^R e^{-t^2}\,dt.$$

Now,

$$|I_2| = \Big| \int_{\gamma_2} e^{iz^2}\, dz \Big| = \Big| \int_0^{\pi/4} e^{iR^2 e^{i2\theta}}\, Ri\, e^{i\theta} d\theta \Big|$$

$$\leq R \int_0^{\pi/4} \Big| e^{iR^2(\cos 2\theta + i \sin 2\theta)} \Big|\, d\theta = R \int_0^{\pi/4} e^{-R^2 \sin 2\theta}\, d\theta.$$

It is known that $\sin 2\theta \geq 4\theta/\pi$ **(show this)** and therefore

$$|I_2| \leq R \int_0^{\pi/4} e^{-R^2 4\theta/\pi}\, d\theta = \frac{\pi}{4R} (1 - e^{-R^2}) \to 0, \quad R \to \infty.$$

Therefore

$$\int_0^{\infty} e^{ix^2}\, dx = \lim_{R\to\infty} \int_0^{R} e^{ix^2}\, dx = \lim_{R\to\infty} I_3 = \lim_{R\to\infty} \frac{1+i}{\sqrt{2}} \int_0^{R} e^{-t^2}\, dt = \frac{(1+i)\sqrt{\pi}}{2\sqrt{2}},$$

where we used $\int_0^\infty e^{-t^2}\, dt = \sqrt{\pi}/2$.

Finally comparing real parts we obtain

$$\int_0^{\infty} \cos(x^2)\, dx = \sqrt{\pi/8}.$$

4) On the circle $|z| = 3/2$, $|z^5| = 243/32$ and $|15z+1| \geq 15|z| - 1 = 21.5$. Thus $|15z+1| > |z|^5$. Hence there is no zero of the polynomial on the circle. If we now denote by $f(z) = 15z + 1$ and by $g(z) = z^5$, then by Rouche's Theorem we have $N(f+g) = N(f)$ inside $|z| = 3/2$. Since the equation $f(z) = 15z + 1 = 0$ has one solution $z_0 = -1/15$, we conclude that $z^5 + 15z + 1$ has one zero inside the circle $|z| < 3/2$.

On the circle $|z| = 2$, $|z^5| = 32$ and $|15z+1| \leq 15|z| + 1 = 31$. Hence there is no zero of the polynomial on the circle and by Rouche's Theorem $N(z^5 + 15z + 1) = N(z^5) = 5$ inside $|z| = 2$. Thus we deduce that in the annulus $\{z : 3/2 < |z| < 2\}$ there are four zeros.

5) Let us split the function $w(z) = f(z) + g(z) = z^{100} + 8z^{10} - 3z^3 + z^2 + z + 1$ such that

$$f(z) = 8z^{10} \qquad \text{and} \qquad g(z) = z^{100} - 3z^3 + z^2 + z + 1.$$

Then for $|z| = 1$ we have

$$|f(z)| = 8 > 7 = |z^{100}| + |3z^3| + |z^2| + |z| + 1 \geq |z^{100} - 3z^3 + z^2 + z + 1|.$$

Therefore the number of solutions of the equation $w(z) = 0$ inside the unit disc coincides with the number of solutions of $z^{10} = 0$, namely 10.

6)
a) Let us consider the case $z : |z| = 1$ and split the function $w(z) = 3z^9 + 8z^6 + z^5 + 2z^3 + 1$ as $f(z) = 8z^6$ and $g(z) = 3z^9 + z^5 + 2z^3 + 1$. Then

$$|f(z)| = 8 > 7 = |3z^9| + |z^5| + |2z^3| + 1 \geq |3z^9 + z^5 + 2z^3 + 1| = |g(z)|.$$

Therefore inside the unit disk there are 6 zeros of w.
b) Let us consider first the case $z : |z| = 2$. Denote $f(z) = 3z^9$ and $g(z) = 8z^6 + z^5 + 2z^3 + 1$. Then

$$\begin{aligned}|f(z)| = 32^9 = 1536 > 512 + 32 + 16 + 1 &= 8|z^6| + |z^5| + 2|z^3| + 1 \\ &\geq |8z^6 + z^5 + 2z^3 + 1| = |g(z)|.\end{aligned}$$

Therefore there are 9 roots of the equation $w(z) = 0$ inside the disc $|z| = 2$. Note that there are no roots of the equation $w(z) = 0$ on the circle $|z| = 1$. Therefore we conclude that there are 3 roots of the equation $w(z) = 0$ in annulus $\{z : 1 < |z| < 2\}$.

7) On the circle $|z| = 1$ we have $|az^n| = |a|$ and $|e^z| = e^{\cos\theta} < e$. Thus $|az^n| > |e^z|, |z| = 1$. The function $az^n - e^z$ has no roots on $|z| = 1$ and no poles. By Rouche's Theorem, $N(az^n - e^z) = N(az^n) = n$.

8) Let us first prove that if $|p(e^{i\theta})| \leq 1$, then $p(z) = z^n$. Indeed, consider

$$q(z) = z^n p(1/z) = 1 + a_{n-1}z + \cdots + a_0 z^n.$$

By using the maximum modulus principle we obtain

$$\max_{|z|\leq 1} |q(z)| = \max_{|z|=1} |q(z)| = \max_{|z|=1} |e^{in\theta} p(e^{-i\theta}| \leq 1,$$

where we also have used the assumption $|p(e^{-i\theta}| \leq 1$. This implies

$$a_{n-1} = \cdots = a_0 = 0$$

and thus $p(z) = z^n$.

9) Assume that such a function exists. Since it does not vanish we have $|(f(z))^{-1}| = e^{-|z|} \leq 1$. However $|f(0)| = 1$ and therefore by the maximum modulus principle we have that f is constant. The constant function cannot satisfy $|f(z)| = e^{|z|}$.

10) Consider the function $g(z) = f(z)/z$. Since f is holomorphic in $\mathbb{D}$ and $f(0) = 0$, we conclude that $g(z)$ is holomorphic in $\mathbb{D}$.
Consider g in $D_\rho = \{z : |z| < \rho\}$, where $\rho < 1$. By the maximum modulus principle $|g|$ has its maximum on the boundary $\gamma_\rho = \{z : |z| = \rho\}$. Since $|f(z)| \leq 1$, $z \in \mathbb{D}$, we have

$$|g(z)| = \frac{|f(z)|}{\rho} \leq \frac{1}{\rho}, \qquad \forall z \in \gamma_\rho.$$

Fixing $z \in D_\rho \subset \mathbb{D}$ and letting $\rho \to 1$ we obtain $|g(z)| \leq 1$ and thus $|f(z)| \leq |z|$ for any $z \in \mathbb{D}$.