MATH50001 - Problems Sheet 6

1. Show that if $0 < a < 1$, then

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}.$$

2. Show that

$$\int_{-\infty}^\infty \frac{x-1}{x^5-1}\,dx = \frac{4\pi}{5}\,\sin\frac{2\pi}{5}.$$

3.* Evaluate

$$\int_0^\infty \cos(x^2)\,dx.$$

4. Show that the polynomial $z^5 + 15z + 1$ has precisely four zeros in the annular region $\{z : 3/2 < |z| < 2\}$.

5. Let $w(z) = z^{100} + 8z^{10} - 3z^3 + z^2 + z + 1$. How many zeros (counting multiplicities) does w has in the unit disc.

6. How many zeros has the complex polynomial $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ in the annulus $\{z : 1 < |z| < 2\}$?

7. Let $a > e$. Show that the equation $e^z = az^n$ has n roots inside the circle $|z| < 1$.

8.* Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Prove that either $p(z) = z^n$, or there is a point z_0, $|z_0| = 1$, such that $|p(z)| > 1$.
[Hint: Use the maximum modulus principle and the fact that $q(z) = z^n p(1/z)$ is also a polynomial of degree n].

9.* Is there a holomorphic function f in the open unit disc and such that $|f(z)| = e^{|z|}$?

10.* Prove Schwarz's Lemma: If f is holomorphic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| \le 1$, then $|f(z)| \le |z|$ for all $z \in \mathbb{D}$.