GROUPS AND RINGS 2021. PROBLEM SHEET 1

QUESTIONS BY ALEXEI N. SKOROBOGATOV

1. Determine the groups $\mathrm{Aut}(\mathbb{Z})$ and $\mathrm{Aut}(\mathbb{Z}/n)$. (Hint: $\mathrm{Aut}(G)$ sends a generator of a cyclic group to another generator. In the second question you can start with the case when n is a prime number.)

2. (a) In lectures we proved that conjugations by the elements of G form a subgroup of $\mathrm{Aut}(G)$. It is denoted by $\mathrm{Inn}(G)$ and is called the subgroup of *inner* automorphisms of G. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
(b) Determine $\mathrm{Inn}(S_3)$ and $\mathrm{Inn}(S_4)$.

3. Let $H \subset G$ be a subgroup.
(a) Prove that if $gH = Hg$ for every $g \in G$, then H is a normal subgroup.
(b) Recall that if the number of cosets $G/H = \{gH | g \in G\}$ is finite, then this number is called the *index* of H in G. Prove that every subgroup of index 2 is normal. Give an example of an index 3 subgroup which is not normal.

4. List all subgroups of the following groups and determine which of them are normal. For each normal subgroup describe the quotient group.
$\mathbb{Z}$, C_n (the cyclic group of order $n \geqslant 2$), S_3, D_8 (the dihedral group of order 8).

5. Let G be a group with centre $Z(G)$ such that the quotient $G/Z(G)$ is a cyclic group. Prove that G is abelian.

6. Let $G = A \times B$ be the product of groups A and B. In lectures we identified A and B with their images in G under the natural injective maps $i_A(a) = (a, e_B)$ and $i_B(b) = (e_A, b)$.
(a) Prove that $G/A \cong B$ and $G/B \cong A$.
(b) Let $A_1 \subset A$ and $B_1 \subset B$ be normal subgroups. Prove that $A_1 \times B_1$ is a normal subgroup of G. Prove that $G/(A_1 \times B_1) \cong (A/A_1) \times (B/B_1)$.
(c) Prove that $Z(A \times B) = Z(A) \times Z(B)$.

7. Prove that every subgroup of a group G that contains $[G, G]$ is normal in G.

8. Describe all pairs of positive integers (m, n) that $C_{mn} \cong C_m \times C_n$.

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