GROUPS AND RINGS 2021. PROBLEM SHEET 2

QUESTIONS BY ALEXEI N. SKOROBOGATOV

1. Let $n \geqslant 3$. Prove that the symmetric group S_n is generated by the following two elements: the 2-cycle (12) and the n-cycle $(12 \dots n)$.

2. Let $n \geqslant 3$. Prove that the alternating group A_n is generated by 3-cycles.

3. Let G be a finite group of even order. Without using Cauchy's theorem, show that G contains an element of order 2.

4. Let $\mathrm{SL}(2, \mathbb{Z})$ be the group of matrices with entries in $\mathbb{Z}$ and determinant 1. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Determine the orders of A, B, and AB, hence conclude that a product of two elements of finite order can have infinite order. Can this happen in an abelian group?

5. Prove that the following groups are not finitely generated:
(a) the group of rational numbers $\mathbb{Q}$ with respect to addition;
(b) the multiplicative group of rational numbers $\mathbb{Q}^\times$;
(c) the quotient group $\mathbb{Q}/\mathbb{Z}$ of $\mathbb{Q}$ (with respect to addition) by the subgroup $\mathbb{Z} \subset \mathbb{Q}$;
(d) the subgroup of $\mathbb{Q}$ (with respect to addition) consisting of the fractions such that the denominator is a power of 2.

6. Prove that every group of order 4 is abelian, hence isomorphic to C_4 or $C_2 \times C_2$.

7. Let G be an abelian p-group and let m be an integer such that $1 \leqslant m \leqslant n$, where $|G| = p^n$. Prove that G contains a subgroup of order p^m. (Hint: Use that every finitely generated abelian group is isomorphic to a product of cyclic groups. This will come up later in the course as Theorem 3.8.)

8. How many isomorphism classes of abelian groups of order p^n are there, for $n = 1, 2, 3, 4, 5$?

9. Prove that the alternating group A_5 is simple. (Hint: Use the fact that a normal subgroup H of a group G is a disjoint union of conjugacy classes $\{gxg^{-1} | g \in G\}$, for some $x \in G$. Compute the cardinalities of all conjugacy classes in A_5, and use the fact that the order of a subgroup of A_5 divides $|A_5| = 60$.)

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