GROUPS AND RINGS 2021. PROBLEM SHEET 3

QUESTIONS BY ALEXEI N. SKOROBOGATOV

1. Prove that the cardinality of the conjugacy class in S_n consisting of the permutations with cycle shape $2^{m_2}3^{m_3}\ldots$ is $n!/\prod_{i=1}^n i^{m_i}m_i!$, where $m_1 = n - \sum_{i=2}^n im_i$.

2. Prove that the set $V = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4. Show that $V \cong C_2 \times C_2$. Consider the action of S_4 on V by conjugations. Determine the orbits of this action and the stabiliser of each point of V.

3. Let G be an abelian group. Let $G_{\rm tors}$ be the torsion subgroup of G. Prove that the torsion subgroup of $G/G_{\rm tors}$ is $\{0\}$.

4. Let m and n be non-negative integers. Prove that $\mathbb{Z}^m \cong \mathbb{Z}^n$ implies $m = n$. (Proceed as follows. Call this group G. Let $2G = \{2x|x \in G\}$. Prove that $\mathbb{Z}^m/2\mathbb{Z}^m$ is isomorphic to the product of m copies of the cyclic group of order 2. Conclude by counting the number of elements in $G/2G$.)

5. Let A be an $n \times n$-matrix with entries in $\mathbb{Z}$ such that $\det(A) \neq 0$. Prove that the quotient group $\mathbb{Z}^n/A\mathbb{Z}^n$ is a finite abelian group with $|\det(A)|$ elements. (Hint: use the Smith normal form.)

6. Let $a_1, \ldots, a_n$ be integers. Let H be the subgroup of $\mathbb{Z}^n$ generated by $(a_1, \ldots, a_n)$. Determine the rank and the torsion subgroup of $\mathbb{Z}^n/H$.

7. Let G be a finite abelian group. Prove that G is not cyclic if and only if there is a prime p such that G contains a subgroup isomorphic to $C_p \times C_p$. (Hint: use the fact that any finite abelian group is isomorphic to a product of finitely many cyclic groups of prime power order.)

8. Let S be a finite subset of a group G such that $a, b \in S$ implies $ab \in S$. Prove that S is a subgroup of G. Give an example of an infinite subset S of a group G such that $a, b \in S$ implies $ab \in S$ but S is not a subgroup of G.

TODO: add Assessed Coursework.

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