GROUPS AND RINGS 2021. PROBLEM SHEET 4

QUESTIONS BY ALEXEI N. SKOROBOGATOV

1. Which of the following are rings? Which are integral domains?
(1) The set of rationals a/b with $a, b \in \mathbb{Z}$ and b odd (usual $+$, $\times$).
(2) The set of rationals a/b with $a, b \in \mathbb{Z}$ and b a power of 2 (usual $+$, $\times$).
(3) $\mathbb{Z}$, with new addition $\oplus$ and multiplication $\otimes$ defined by

$$m \oplus n = m + n + 2 \text{ and } m \otimes n = mn + 2m + 2n + 2.$$

2. Let R be a ring. Deduce directly from the axioms of a ring that for any $x, y \in R$ we have $(-x)(-y) = xy$.

3. Let $F = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.
(1) Prove that F is a field.
(2) Prove that $\mathbb{Q}$ has exactly one subfield (namely $\mathbb{Q}$ itself).
(3) Prove that F has exactly two subfields.

4. Prove that $\mathbb{Q}$ contains infinitely many subrings which are integral domains.

5. (Quaternions) Let $\mathbb{H}$ be the set of 2×2 matrices

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : z, w \in \mathbb{C} \right\}.$$

Prove that $\mathbb{H}$ is a division ring.

6. Prove that if F_1 and F_2 are subfields of a field K then $F_1 \cap F_2$ is a subfield of K.

7. Let I and J be ideals of a commutative ring R. Define

$$I + J = \{a + b : a \in I \text{ and } b \in J\}.$$

Prove that $I + J$ is an ideal of R.

8. Suppose that F is a finite field with p^n elements. Prove that $r^{p^n} = r$ for all $r \in F$.

9. Let R be a ring in which $x^2 = x$ for all $x \in R$. Prove that R is commutative.

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