Groups and Rings: Unseen Problem Sheet 1

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Question 1. Let G and H be groups. Denote by $\mathrm{Hom}(G, H)$ the group of homomorphisms from G to H (the operation being the group operation in H). Describe the following groups:

1. $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z})$
2. $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$
3. $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$

In the above, you should consider $\mathbb{Q}$ and $\mathbb{Z}$ with the operation being addition. $\mathbb{C}^\times$ has multiplication as its operation.

The first two groups are trivial.

Suppose $f : \mathbb{Q} \longrightarrow \mathbb{Z}$ is a nontrivial homomorphism - that is, it sends some $q \in \mathbb{Q}$ to some nonzero element $f(q) \in \mathbb{Z}$. There exists some prime number p that doesn't divide $f(q)$, so there is no element m of $\mathbb{Z}$ such that $mp = f(q)$. However, q/p is a well-defined element of $\mathbb{Q}$, and since homomorphisms preserve the group structure, $f(q/p)$ must be an element of $\mathbb{Z}$ such that $pf(q/p) = f(p(q/p)) = f(q)$: contradiction.

Now suppose $f : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}$ is a nontrivial homomorphism. There is a nonzero $m \in \mathbb{Z}/n\mathbb{Z}$ whose image $f(m)$ is nonzero in $\mathbb{Z}$. We know $nm \cong 0 \in \mathbb{Z}/n\mathbb{Z}$ and so we must have $nf(m) = 0$ in $\mathbb{Z}$. But there no nonzero elements of $\mathbb{Z}$ with finite order, so we have a contradiction.

The third group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Since any group homomorphism $f : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^\times$ must have the property that

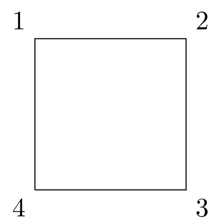
$$f(mx) = f(x)^m$$

it follows that any $x \in \mathbb{Z}/n\mathbb{Z}$ must be mapped to an nth root of unity in $\mathbb{C}^\times$, and also that f is determined by where the generator $1 \in \mathbb{Z}/n\mathbb{Z}$ is mapped to. $f(1)$ can be any nth root of unity, so $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$ is in bijection with the set $\{\zeta \in \mathbb{C}^\times | \zeta^n = 1\}$. The group operation on $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$ is given by the multiplicative group operation on $\mathbb{C}^\times$, so

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \cong \{\zeta \in \mathbb{C}^\times | \zeta^n = 1\} \cong \mathbb{Z}/n\mathbb{Z}$$

Question 2. Let H be a normal subgroup of G and K be a normal subgroup of H. Is K a normal subgroup of G?

K is not necessarily a normal subgroup of G. Since all subgroups of an abelian group are normal, we know our counterexample will have to be nonabelian. The trivial subgroup is also always normal, so K in our counterexample will have to have order at least 2. Let's try to use the fact (from Problem Sheet 1) that all subgroups of index 2 are normal. So we want K to have order 2, H to have order 4, and G to have order 8, with G nonabelian. D_8, the dihedral group of order 8 is a nonabelian group of order 8. We can think of it as the group of symmetries of a square by reflections and rotations. You can write the elements as permutations by numbering the vertices of the square like so:



We want K to be of order 2, so it either has to be the subgroup generated by the rotation by 180° or a subgroup generated by a reflection. We want K to be such that $gK \neq Kg$ for some $g \in G$. We know that reflection in the vertical centre line, (12)(34), does not commute with rotation by 90°, (1234):

$$(12)(34)(1234) = (24)$$
$$(1234)(12)(34) = (31)$$

Therefore if we set $K = \{e, (12)(34)\}$, then K is not normal in G. This reflection does however commute with the rotation by 180°:

$$(12)(34)(13)(24) = (23)(14) = (13)(24)(12)(34)$$

The group generated by (12)(34) and (13)(24) has order 4, so is normal in D_8. Set $H = \{e, (12)(34), (13)(24), (14)(23)\}$. Since H has index 2 in G, it is normal, and K is normal in H.

Question 3. Let H be a subgroup of G such that $g^2 \in H$ for any $g \in G$.
Prove H is a normal subgroup of G.

We want to show that for every $h \in H$ and $g \in G$ we have $ghg^{-1} \in H$. We can re-express ghg^{-1} in terms of h and squares as follows:

$$ghg^{-1} = ghgg^{-1}g^{-1} = ghg(g^{-1})^2 = ghghh^{-1}(g^{-1})^2 = (gh)^2h^{-1}(g^{-1})^2.$$

Question 4. Prove that an infinite group G has infinitely many subgroups.

We can write down an infinite list of subgroups of G by simply taking the cyclic group generated by each element of G, of which there are infinitely many. We just have to prove that these are in fact infinitely many distinct subgroups, and not just finitely many subgroups expressed with infinitely many different generators.

Suppose for a contradiction that only a finite number of these cyclic subgroups are distinct. Since we know that every $g \in G$ is in one of these subgroups, we have that the finite union of these subgroups is equal to G. If all of these subgroups had finite order, then G would be finite, which is a contradiction. So let's assume that one of these cyclic subgroups H has infinite order, and let h be a generator for H. We can construct infinitely many cyclic subgroups of H: let $K_i = \langle h^i \rangle$, where $i \in \mathbb{N}$. We know that these are in fact infinitely many distinct groups, because if $i < j$ then $h^i \notin K_j$.

Question 5. Prove the following statement or find a counterexample.

Let H and K be subgroups of a group G and L be a normal subgroup of G. If $HL = KL$ and $H \cap L = K \cap L$, then $H = K$.

The statement is false. It's quite hard to see, because the statement $H \cong K$ is actually true. Here's a counterexample: let $G = S_3$, $H = \{e, (12)\}$, $K = \{e, (13)\}$ and $L = \{e, (123), (132)\}$. Then $HL = KL = S_3$ and $H \cap L = K \cap L = \{e\}$, but $H \neq K$.