UNSEEN PROBLEM SHEET 2

Note: The definitions introduced in Question 3 below are *not examinable*, unless they also appear in the lecture notes. Comments or corrections to `wl4714@ic.ac.uk`

1) Let G be a group such that every element $g \in G, g \neq e$ has order 2. Prove that G must be abelian.

Solution: Let $x, y \in G$ be arbitrary. We need to show that $xy = yx$. Since xy has order dividing 2 we have

$$(xy)^2 = e \implies xy = y^{-1}x^{-1}$$

Since x and y have order dividing 2 we have $x = x^{-1}, y = y^{-1}$ giving $xy = yx$ as required.

2) Let $\mathrm{GL}_2(\mathbb{R})$ be the group of invertible 2×2 matrices with coefficients in $\mathbb{R}$. Show that

$$\mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (A, v) \mapsto Av$$

defines an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$.

- What are the orbits of that action?
- What are the fixed points of that action?
- What is the stabilizer of $(1, 0) \in \mathbb{R}^2$?

Consider now the subgroup

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \mid 0 \leq \phi < 2\pi \right\}$$

Equivalently, $\mathrm{SO}_2(\mathbb{R})$ is the group $\{A \in \mathrm{GL}_2(\mathbb{R})\colon AA^T = I,\ \det(A) = 1\}$. SO(2) also acts on $\mathbb{R}^2$ via the same operation.

- What are the orbits of that action?
- What are the fixed points of that action?
- What is the stabilizer of $(1, 0) \in \mathbb{R}^2$?

Solution: There are two orbits of the action of $\mathrm{GL}_2(\mathbb{R})$: $\{(0, 0)\}$ and $\mathbb{R}^2 \setminus (0, 0)$. To see the latter, note that for $v \in \mathbb{R}^2, v \neq 0$ by linear algebra we can find a matrix A mapping v to, say, $(1, 0)$ so that any nonzero vector is in the orbit of $(1, 0)$.
A fixed point v of the action must satisfy $Av = v$ for all $A \in \mathrm{GL}_2(\mathbb{R})$. Taking $A = -I$, this gives $v = -v$, which means that $v = (0, 0)$ is the only fixed point of the action.
The stabilizer of $(1, 0)$ consists of all matrices of the form

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $b \neq 0$.
The orbits of the action of $\mathrm{SO}_2(\mathbb{R})$ are circles ∂B_r of fixed radius r around the origin. To see this, recall from linear algebra that we have $\|Av\| = \|v\|$ for all $A \in \mathrm{SO}_2(\mathbb{R})$ which implies that Av lies on the same circle as v. Moreover, given a point $v \in \partial B_r$, write v in polar coordinates as (r, θ). Then the matrix

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

maps $(0, r)$ to v, so that every point $v \in \partial B_r$ lies in the orbit of $(0, r)$.
The fixed points of the action are the same as above.
The stabilizer of $(1, 0)$ consists of all matrices of the form

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

which are also in $\mathrm{SO}_2(\mathbb{R})$, so applying the conditions $AA^T = I$ and $\det A = 1$ shows that the stabilizer consists just of the identity I (geometrically, no nontrivial rotation can fix a nonzero real vector).

3) Let G be a group acting on a set X. For $g \in \mathrm{GL}_2(\mathbb{C})$, define $\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$.

- Show that the set $H = \{g \in G : \mathrm{Fix}(g) = X\}$ is a subgroup of G.
- If H is normal, show that there is an induced well-defined action of G/H on X.

Let $\mathrm{GL}_2(\mathbb{C})$ be the group of of invertible 2×2 matrices with coefficients in $\mathbb{C}$, and define the *Riemann sphere* $\mathbb{CP}^1$ as the set-theoretic union of $\mathbb{C}$ and the singleton set $\{\infty\}$. Define an action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{CP}^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

with the understanding that $\frac{-d}{c}$ maps to ∞, and ∞ maps to $\frac{a}{c}$ (If $c = 0$, then ∞ maps to ∞).

- Verify that this is indeed a group action.
- Determine the subgroup $H = \{A \in \mathrm{GL}_2(\mathbb{C}) : \mathrm{Fix}(A) = \mathbb{CP}^1\}$, and check that H is a normal subgroup of $\mathrm{GL}_2(\mathbb{C})$. Deduce that $\mathrm{GL}_2(\mathbb{C})/H$ acts on $\mathbb{CP}^1$ as well. (The quotient $\mathrm{GL}_2(\mathbb{C})/H$ is called the *projective general linear group* $\mathrm{PGL}_2(\mathbb{C})$, and is important in geometry and algebra.)
- For $A \notin H$, show that $\mathrm{Fix}(A)$ consists of two elements unless $(a-d)^2 + 4bc = 0$, in which case $\mathrm{Fix}(A)$ consists of one element.
- Show that the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{CP}^1$ is *sharply* 3*-transitive*: This means that for every two pairwise distinct triples $(z_1, z_2, z_3), (w_1, w_2, w_3)$ where $z_i, w_i \in \mathbb{CP}^1$, there exist a unique $A \in \mathrm{PGL}_2(\mathbb{C})$ such that $A \cdot z_i = w_i$ for all $1 \le i \le 3$. (Hint: Show first that given such a triple (z_1, z_2, z_3), we can find a unique $A \in \mathrm{PGL}_2(\mathbb{C})$ mapping z_1 to 0, z_2 to 1, and z_3 to ∞)

Solution: I am assuming that G acts on the left, the argument for a right action is the same. The verification that H is a subgroup is standard. The induced action of G/H is given by $[g] \cdot z = gz$. This is well-defined since any other representative of $[g]$ is given by gh for some $h \in H$, and $(gh)z = g(hz) = gz$.

It is clear that $I \cdot z = z$. To verify associativity, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Suppose first that $z \neq \infty$, $z \neq -\frac{d'}{c'}$, $Bz \neq -\frac{d}{c}$. We compute

$$A \cdot (B \cdot z) = \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d}$$

and

$$(AB) \cdot z = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} \cdot z = \frac{(aa' + bc')z + ab' + bd'}{(ca' + dc')z + cb' + dd'}$$

These two expressions are the same (unless there is a typo). We still need to check some special values of z:

- If $z = -\frac{d'}{c'}$ then $A \cdot (B \cdot z) = \frac{a}{c}$ and $(AB) \cdot z = \frac{-(\det B)a}{-(\det B)c} = \frac{a}{c}$
- If $z = \infty$, then $A \cdot (B \cdot z) = \frac{aa'+bc'}{ca'+dc'} = (AB) \cdot z$
- If $Bz = -\frac{d}{c}$, then $z = -\frac{cb'+dd'}{ca'+dc'}$. We compute $A \cdot (B \cdot z) = \infty$ and $(AB) \cdot z = \infty$ as well.

One computes that H consists of matrices of the form aI for $a \neq 0$. Since H is contained in the centre of $\mathrm{GL}_2(\mathbb{C})$, it follows that H is normal. By the first part of the question $\mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbb{CP}^1$ as well.
To compute $\mathrm{Fix}(A)$, we need to solve the equation

$$\frac{az+b}{cz+d} = z$$

If $c \neq 0$ this gives a quadratic equation in z which has two distinct roots iff $(a-d)^2 + 4bc = 0$. If $c = 0$, we get the equation $\frac{a}{d}z + \frac{b}{d} = z$. If $a \neq d$ this equation has one complex solution $z = \frac{b}{d-a}$, and the other fixed point is ∞ (Recall that $A \cdot \infty = \infty$ iff $c = 0$). Note that $(a-d)^2 + 4bc \neq 0$ in this case. If $a = d$, the equation has no complex solution, and ∞ is the unique solution. Note that $(a-d)^2 + 4bc = 0$ in this case.

For the last part, suppose first that z_1, z_2, z_3 are all not equal to ∞. We follow the hint and want to first find $A \in \mathrm{GL}_2(\mathbb{C})$ such that $A \cdot z_1 = 0, A \cdot z_2 = 1, A \cdot z_3 = \infty$. This gives us the equations

$$az_1 + b = 0,\ az_2 + b = cz_2 + d,\ cz_3 + d = 0$$

This expresses d, b in terms of a, c, and gives $a(z_2 - z_1) = c(z_2 - z_3)$. Since the z_i are distinct, we can solve this to get $c = \frac{a(z_2 - z_1)}{z_2 - z_3}$. Hence there is a one-parameter family of matrices satisfying the three equations, given by

$$A_a = \begin{pmatrix} a & -az_1 \\ \frac{a(z_2-z_1)}{z_2-z_3} & -\frac{a(z_2-z_1)z_3}{z_2-z_3} \end{pmatrix}$$

It follows that there is a unique class in $\mathrm{PGL}_2(\mathbb{C})$ satisfying the three equations above, a representative is given for example by the matrix A_1.

If one of the z_i is ∞, we argue similarly. I will only do the case where $z_1 = \infty$. This gives the equations

$$a = 0,\ b = cz_2 + d,\ cz_3 + d = 0$$

(Remember that we defined $A \cdot \infty = \frac{a}{c}$) so we get a 1-parameter family of matrices

$$A_c = \begin{pmatrix} 0 & c(z_2 - z_3) \\ c & -cz_3 \end{pmatrix}$$

in $\mathrm{GL}_2(\mathbb{C})$, and we again get a unique class in $\mathrm{PGL}_2(\mathbb{C})$.

Note: You may have noticed that this solution is quite awkward and requires to split into many different cases. The question can be solved in a much nicer and uniform way by introducing *homogeneous coordinates* on $\mathbb{CP}^1$. The process of adding ∞ to $\mathbb{C}$ to obtain $\mathbb{CP}^1$ is called *compactification*, and the reason for doing so is that compact spaces have much nicer properties, they are in a certain way similar to finite sets (this is intentionally vague). If you want to learn more about this, the third year course Algebraic Curves covers this and much more beautiful mathematics!