UNSEEN PROBLEM SHEET 2

1) Let G be a group such that every element $g \in G, g \neq e$ has order 2. Prove that G must be abelian.

2) Let $\mathrm{GL}_2(\mathbb{R})$ be the group of invertible 2×2 matrices with coefficients in $\mathbb{R}$. Show that

$$\mathrm{GL}_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (A, v) \mapsto Av$$

defines an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2$.

- What are the orbits of that action?
- What are the fixed points of that action?
- What is the stabilizer of $(1, 0) \in \mathbb{R}^2$?

Consider now the subgroup

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \mid 0 \leq \phi < 2\pi \right\}$$

Equivalently, $\mathrm{SO}_2(\mathbb{R})$ is the group $\{A \in \mathrm{GL}_2(\mathbb{R}) \colon AA^T = I,\ \det(A) = 1\}$. SO(2) also acts on $\mathbb{R}^2$ via the same operation.

- What are the orbits of that action?
- What are the fixed points of that action?
- What is the stabilizer of $(1, 0) \in \mathbb{R}^2$?

3) Let G be a group acting on a set X. For $g \in \mathrm{GL}_2(\mathbb{C})$, define $\mathrm{Fix}(g) = \{x \in X \colon g \cdot x = x\}$.

- Show that the set $H = \{g \in G \colon \mathrm{Fix}(g) = X\}$ is a subgroup of G.
- If H is normal, show that there is an induced well-defined action of G/H on X.

Let $\mathrm{GL}_2(\mathbb{C})$ be the group of of invertible 2×2 matrices with coefficients in $\mathbb{C}$, and define the *Riemann sphere* $\mathbb{CP}^1$ as the set-theoretic union of $\mathbb{C}$ and the singleton set $\{\infty\}$. Define an action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{CP}^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

with the understanding that $\frac{-d}{c}$ maps to ∞, and ∞ maps to $\frac{a}{c}$ (If $c = 0$, then ∞ maps to ∞).

- Verify that this is indeed a group action.
- Determine the subgroup $H = \left\{A \in \mathrm{GL}_2(\mathbb{C}) \colon \mathrm{Fix}(A) = \mathbb{CP}^1\right\}$, and check that H is a normal subgroup of $\mathrm{GL}_2(\mathbb{C})$. Deduce that $\mathrm{GL}_2(\mathbb{C})/H$ acts on $\mathbb{CP}^1$ as well. (The quotient $\mathrm{GL}_2(\mathbb{C})/H$ is called the *projective general linear group* $\mathrm{PGL}_2(\mathbb{C})$, and is important in geometry and algebra.)
- For $A \notin H$, show that $\mathrm{Fix}(A)$ consists of two elements unless $(a - d)^2 + 4bc = 0$, in which case $\mathrm{Fix}(A)$ consists of one element.
- Show that the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{CP}^1$ is *sharply* 3*-transitive*: This means that for every two pairwise distinct triples $(z_1, z_2, z_3), (w_1, w_2, w_3)$ where $z_i, w_i \in \mathbb{CP}^1$, there exist a unique $A \in \mathrm{PGL}_2(\mathbb{C})$ such that $A \cdot z_i = w_i$ for all $1 \leq i \leq 3$. (Hint: Show first that given such a triple (z_1, z_2, z_3), we can find a unique $A \in \mathrm{PGL}_2(\mathbb{C})$ mapping z_1 to 0, z_2 to 1, and z_3 to ∞)