Groups and Rings

Unseen Problem Sheet 3

October 28, 2021

Q1. Prove that, for every prime p, there are, up to isomorphism, precisely two nonisomorphic groups of order p^2; these are C_{p^2} and $C_p \times C_p$.

Q2. Let G be a finite abelian group and let p be a prime number. If p divides $|G|$, and N_p denotes the number of elements of G of order p, then

$$N_p \equiv -1 (\mathrm{mod}\ p)$$

(Hint : Use Cauchy's theorem.)

Q3. Let H be a subgroup of index p in the finite group G, where p is the smallest prime divisor of $|G|$. Prove that H is a normal subgroup of G.

Q4. Let G be a group of order n. Prove that :
(1) Let $g_1, g_2, \cdots \in G$ be such that $g_1 \neq e$ and $H_i \subsetneq H_{i+1}$, where H_i is the subgroup generated by $g_1, \ldots, g_i$. Then $|H_i| \geq 2^i$ for each i.
(2) G can be generated by at most $log_2 n$ elements.
Use Cayley's Theorem to conclude that the number of non-isomorphic groups of order n does not exceed $(n!)^{log_2 n}$.