GROUPS AND RINGS
RINGS UNSEEN PROBLEMS SHEET 1
SOLUTIONS

1) Consider the set $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that f vanishes at $\pm$infinity, that is,

$$\lim_{t\to\pm\infty} f(t) = 0.$$

   a) Prove that $C_0(\mathbb{R})$ is an Abelian group with respect to the ordinary sum of functions.
   Answer: The addition of two continuous functions is continuous. By properties of limits, if $f, g \in C_0(\mathbb{R})$ then

$$\lim_{t\to\pm\infty} (f + g) = 0$$

   also so $C_0(\mathbb{R})$ is closed under addition. The zero function $f \equiv 0$ is the additive identity, and $-f$ is the additive inverse of f (note $-f$ is continuous and converges to zero at $\pm\infty$ if f does). Associativity and commutativity follow from the same properties for the group $\mathbb{R}$ itself.

   b) Prove that the multiplication of two functions in $C_0(\mathbb{R})$ is in $C_0(\mathbb{R})$. Does multiplication of functions distribute over addition of functions in $C_0(\mathbb{R})$?
   Answer: By properties of limits, if $f, g \in C_0(\mathbb{R})$ then

$$\lim_{t\to\pm\infty} (fg) = 0$$

   and certainly fg is continuous. Since multiplication distributes over addition in $\mathbb{R}$ for every x we have $f(x)(g(x)+h(x)) = f(x)g(x)+f(x)h(x)$ and so $f(g + h) = fg + fh$ in $C_0(\mathbb{R})$.

   c) Does $C_0(\mathbb{R})$ have a multiplicative identity element?
   Answer: No. The multiplicative identity in $\mathbb{R}$ is 1, but the continuous function $f \equiv 1$ is not contained in $C_0(\mathbb{R})$ because the limit as $t \to \pm\infty$ is 1, not 0. Thus $C_0(\mathbb{R})$ satisfies all the axioms of a ring except having a multiplicative identity.

2) A ring does not necessarily need to have a multiplicative identity element. Let us call a ring without identity a *rng* (since we have "removed the i"). Alternatively one may refer to a ring with identity as a *unital ring*.

   a) If R is a rng, prove that the Abelian group $\hat{R} := R \oplus \mathbb{Z}$ with the multiplication

$$(r, n) \cdot (s, m) = (rs + mr + ns, nm)$$

   is a ring with identity element. What is the multiplicative identity of $\hat{R}$?
   Answer: It is immediate that $\hat{R}$ satisfies the axioms of an Abelian group with respect to addition. We need to check that $\hat{R}$ satisfies associativity and distributivity of multiplication, and that it has an identity element.

If $(r_1, n_1), (r_2, n_2), (r_3, n_3)$ are elements of $\hat{R}$, then

$$\begin{aligned}
&((r_1, n_1) \cdot (r_2, n_2)) \cdot (r_3, n_3) \\
&= (r_1 r_2 + n_2 r_1 + n_1 r_2, n_1 n_2) \cdot (r_3, n_3) \\
&= ((r_1 r_2 + n_2 r_1 + n_1 r_2) r_3 + n_3 (r_1 r_2 + n_2 r_1 + n_1 r_2) + n_1 n_2 r_3, n_1 n_2 n_3) \\
&= (r_1 (r_2 r_3 + n_2 r_3 + n_3 r_2) + n_2 n_3 r_1 + n_1 (r_2 r_3 + n_2 r_3 + n_3 r_2), n_1 n_2 n_3) \\
&= (r_1, n_1) \cdot ((r_2, n_2) \cdot (r_3, n_3)).
\end{aligned}$$

Thus multiplication in $\hat{R}$ is associative. We also have

$$\begin{aligned}
&(r_1, n_1) \cdot ((r_2, n_2) + (r_3, n_3)) \\
&= (r_1, n_1) \cdot (r_2 + r_3, n_2 + n_3) \\
&= (r_1 (r_2 + r_3) + (n_2 + n_3) r_1 + n_1 (r_2 + r_3), n_1 (n_2 + n_3)) \\
&= ((r_1 r_2 + n_2 r_1 + n_1 r_2) + (r_1 r_3 + n_3 r_1 + n_1 r_3), (n_1 n_2) + (n_1 n_3)) \\
&= (r_1, n_1) \cdot (r_2, n_2) + (r_1, n_1) \cdot (r_3, n_3).
\end{aligned}$$

Thus multiplication is distributive over addition. Finally note that for any $(r, n) \in \hat{R}$, we have

$$(0, 1) \cdot (r, n) = (1r, 1n) = (r, n) = (r, n) \cdot (0, 1)$$

so $(0, 1) \in \hat{R}$ is the multiplicative identity $1_{\hat{R}}$.

b) Define a rng homomorphism $f : R \to S$ by removing the axiom that $f(1_R) = 1_S$ from the definition of a ring homomorphism. Prove that the map $f : R \to \hat{R}$ defined by

$$f(r) = (r, 0)$$

is an injective rng homomorphism from R to $\hat{R}$.

Answer: The map is obviously a group homomorphism since it is just the inclusion $R \hookrightarrow R \oplus \mathbb{Z}$. We need to check

$$f(r_1 r_2) = (r_1 r_2, 0) = (r_1, 0) \cdot (r_2, 0) = f(r_1) f(r_2)$$

so f is a rng homomorphism. We recall that the additive identity in $\hat{R}$ is $(0, 0)$, so if $f(r) = (0, 0)$ we must have $r = 0$ so f is injective.

c) There is a map $g : \hat{R} \to \mathbb{Z}$ defined by

$$g(r, n) = n.$$

What is the kernel of g? Conclude that every rng is an ideal in a ring.

Answer: Suppose $g(r, n) = 0$. Then $n = 0$ so $\ker g = \operatorname{im} f$ is simply the copy of R inside $R \oplus \mathbb{Z}$. Since g is a ring homomorphism from $\hat{R}$ to $\mathbb{Z}$, its kernel is an ideal. Thus every rng can be realised as an ideal in its unitalization.

d) (**Harder**) Consider the ring $C_0(\mathbb{R})$. We can consider the unitalization of $C_0(\mathbb{R})$ as an algebra, which means we take $\hat{C}_0(\mathbb{R}) := C_0(\mathbb{R}) \oplus \mathbb{R}$ instead of $\oplus \mathbb{Z}$. Let us write $\bar{\mathbb{R}} = \mathbb{R} \cup \{x_\infty\}$. Define a map $\varphi : \hat{C}_0(\mathbb{R}) \to C(\bar{\mathbb{R}})$ by

$$(f, \alpha) \mapsto \left(x \mapsto \begin{cases} f(x) + \alpha & x \in \mathbb{R} \\ \alpha & x = x_\infty \end{cases} \right)$$

for $f \in C_0(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Let $C(S^1)$ denote the ring of continuous functions $h : [-\pi/2, \pi/2] \to \mathbb{R}$ such that $h(-\pi/2) = h(\pi/2)$. Prove that $C(\bar{\mathbb{R}})$ is isomorphic to $C(S^1)$ and conclude that the unitalization of the functions vanishing at infinity on $\mathbb{R}$ is equivalent to the continuous functions on the circle, its one-point compactification. (Hint: construct an isomorphism between functions on $\bar{\mathbb{R}}$ and functions on the interval $[-\pi/2, \pi/2]$ using $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$. Check that under the above constructions, if $f \in C_0(\mathbb{R})$ then the induced function $h \in C(S^1)$ is continuous on $[-\pi/2, \pi/2]$. Show that this map is injective and surjective, and is a ring homomorphism.)

Answer: Define a map $\varphi : \hat{C}_0(\mathbb{R}) \to C(S^1)$ as follows. Given an element (f, α) in $\hat{C}_0(\mathbb{R})$, consider the function $h : [-\pi/2, \pi/2] \to \mathbb{R}$ defined by

$$h(x) = \varphi(f)(x) := \begin{cases} \alpha & x = -\pi/2, \pi/2 \\ f(\tan(x)) + \alpha & -\pi/2 < x < \pi/2 \end{cases}.$$

Since the composition of continuous functions is continuous, h is continuous on $(-\pi/2, \pi/2)$. Let us verify continuity at one endpoint of the interval (the other being similar). Here we use the property of limits that if the limit of a composition of two functions exists then

$$\lim_{x\to c} f(g(x)) = \lim_{u\to b} f(u)$$

where $\lim_{x\to c} g(x) = b$.

$$\begin{aligned} \lim_{x\to\pi/2} h(x) &= \lim_{x\to\pi/2} (f(\tan(x)) + \alpha) \\ &= \lim_{x\to\pi/2} (f(\tan(x))) + \alpha \\ &= \lim_{t\to\infty} f(t) + \alpha \\ &= \alpha \end{aligned}$$

Thus h is continuous at the endpoints $\pi/2$ (and similarly for $-\pi/2$). In particular $h \in C(S^1)$. One can verify by direct computation that $\varphi(f+g)(x) = \varphi(f)(x)+\varphi(g)(x)$ for every $x \in [-\pi/2, \pi/2]$. We also have the zero element $(0, 0) \in \hat{C}_0(\mathbb{R})$ maps to the function $h \equiv 0$ in $C(S^1)$, and the multiplicative identity $(0, 1) \in \hat{C}_0(\mathbb{R})$ maps to the constant function $h \equiv 1$, which is the multiplicative identity in $C(S^1)$. Let us check that φ preserves multiplication. We have

$$\varphi((f, \alpha) \cdot (g, \beta)) = \varphi(fg + \alpha g + \beta f, \alpha\beta).$$

For $x = \pi/2, -\pi/2$ then

$$\varphi((f, \alpha) \cdot (g, \beta))(x) = \alpha\beta = \varphi(f, \alpha)(x)\varphi(g, \beta)(x)$$

and for $x \in (-\pi/2, \pi/2)$ we have

$$\begin{aligned} &\varphi((f, \alpha) \cdot (g, \beta))(x) \\ &= f(\tan(x))g(\tan(x)) + \alpha g(\tan(x)) + \beta f(\tan(x)) + \alpha\beta \\ &= (f(\tan(x)) + \alpha)(g(\tan(x)) + \beta) \\ &= \varphi(f, \alpha)(x)\varphi(g, \beta)(x). \end{aligned}$$

Thus φ is a ring homomorphism from $\hat{C}_0(\mathbb{R})$ to $C(S^1)$. Clearly if $\varphi(f, \alpha)$ is the zero function then $\alpha = 0$ (since it is zero at the endpoints) and

$f(\tan(x)) = 0$ for every $x \in (-\pi/2, \pi/2)$. Since arctan is a bijection from $(-\pi/2, \pi/2) \to \mathbb{R}$, we have $f(t) = 0$ for all $t \in \mathbb{R}$ so φ is injective.
Finally to see φ is surjective, given some $h \in C(S^1)$ then the element $(f, \alpha) \in \hat{C}_0(\mathbb{R})$ defined by $\alpha = h(\pi/2)$ and

$$f(t) = h(\arctan(t)) - \alpha$$

where $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$. Since $\arctan^{-1} = \tan$, this is the inverse construction to φ and hence φ is a ring isomorphism $\hat{C}_0(\mathbb{R}) \to C(S^1)$.

The above is called the *unitalization of a ring*. The process in (d) above is a method to construct the one-point compactification of a space using ring theory.

3) Even if a ring R has a multiplicative identity 1_R, not every element r of R needs to have a multiplicative inverse r^{-1}. A *multiplicative set* $S \subset R$ is a subset of R (not necessarily a subgroup or subring!) which is closed under multiplication and contains 1_R.

a) Define a set $S^{-1}R$ as pairs $(s, r) \in S \times R$ modulo the equivalence relation

$$(s, r) \sim (s', r')$$

if and only if there exists a $t \in S$ such that $t(sr' - s'r) = 0$. Define addition and multiplication in $S^{-1}R$ as

$$[s, r] + [s', r'] := [ss', rs' + r's]$$

$$[s, r] \cdot [s', r'] := [ss', rr'].$$

Prove that these operations are well-defined and that $S^{-1}R$ forms a ring.
Answer: Let us write $[s, r] = r/s$ for the rest of this question. Then the equivalence relation states $(tr)/(ts) = r/s$ for $t \in S$. It follows from properties of fractions that addition and scalar multiplication are well-defined, distributive, and associative. The additive identity is $0/1$ and the multiplicative identity is $1/1$. The additive inverse of r/s is $(-r)/s$

b) Define a map $f : R \to S^{-1}R$ by $f(r) = (1, r)$. Prove that f is a ring homomorphism. Prove that f is injective if and only if S does not contain any zero divisors of R.
Answer: We have $f(r_1r_2) = (r_1r_2)/1 = (r_1/1)(r_2/1) = f(r_1)f(r_2)$. The other properties of a ring homomorphism are also easily satisfied. Finally if $f(r) = 0/1$ then $r/1 = 0/1$. Recall that $r/1 = 0/1$ if and only if there exists a $t \in S$ such that $t(1 \cdot r - 0 \cdot 1) = tr = 0$.
If S has a zero-divisor t, then let $r \neq 0$ be such that $tr = 0$. Then $f(r) = 0/1$ so f is not injective. On the other hand suppose S has no zero-divisors. Then for any r such that $f(r) = 0/1$, we have some $t \in S$ such that $tr = 0$. But since S has no zero-divisors, this implies $r = 0$. Thus f is injective if and only if S has no zero-divisors.

c) Prove that if $s \in S$, then $f(s)$ has a multiplicative inverse in $S^{-1}R$.
Answer: The element $1/s$ is the multiplicative inverse to $s/1 = f(s)$.

d) Suppose $0 \in S$. What is $S^{-1}R$?
Answer: Let $(s, r), (s', r') \in S^{-1}R$. Then $0(sr' - s'r) = 0$ so $(s, r) \sim (s', r')$. Thus $S^{-1}R$ consists of a single equivalence class, which must contain $(1, 0) = 0/1$. Thus $S^{-1}R = \{0\}$ is the zero ring.

e) Suppose R is an integral domain and $S = R\backslash\{0\}$. Prove that $S^{-1}R$ is a field.

Answer: We need to verify that any element of $S^{-1}R$ that isn't zero has a multiplicative inverse. Let $r/s \in S^{-1}R$ be nonzero. Then $r, s \neq 0$. Then we see

$$(r/s)(s/r) = (rs)/(sr).$$

Since R is an integral domain, it is commutative, so

$$(r/s)(s/r) = (rs)/(rs).$$

Since R is an integral domain, $rs \neq 0$ for $r, s \neq 0$, and so $rs \in S$. But then $(rs)/(rs) = 1/1$ in $S^{-1}R$, and so r/s has a multiplicative inverse s/r. Thus every non-zero element of $S^{-1}R$ has an inverse and $S^{-1}R$ is a field.

The construction above is called the *localization of* R *at* S, and elements (s, r) are written $\frac{r}{s}$. The localization $S^{-1}R$ adds in a multiplicative inverse for every element of S. The localization $(R\backslash\{0\})^{-1}R$ is called the *field of fraction* of R.