GROUPS AND RINGS
RINGS UNSEEN PROBLEMS SHEET 1

1) Consider the set $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that f vanishes at $\pm$infinity, that is,
$$\lim_{t \to \pm\infty} f(t) = 0.$$
   a) Prove that $C_0(\mathbb{R})$ is an Abelian group with respect to the ordinary sum of functions.
   b) Prove that the multiplication of two functions in $C_0(\mathbb{R})$ is in $C_0(\mathbb{R})$. Does multiplication of functions distribute over addition of functions in $C_0(\mathbb{R})$?
   c) Does $C_0(\mathbb{R})$ have a multiplicative identity element?

2) A ring does not necessarily need to have a multiplicative identity element. Let us call a ring without identity a *rng* (since we have "removed the i"). Alternatively one may refer to a ring with identity as a *unital ring*.
   a) If R is a rng, prove that the Abelian group $\hat{R} := R \oplus \mathbb{Z}$ with the multiplication
$$(r, n) \cdot (s, m) = (rs + mr + ns, nm)$$
   is a ring with identity element. What is the multiplicative identity of $\hat{R}$?
   b) Define a rng homomorphism $f : R \to S$ by removing the axiom that $f(1_R) = 1_S$ from the definition of a ring homomorphism. Prove that the map $f : R \to \hat{R}$ defined by
$$f(r) = (r, 0)$$
   is an injective rng homomorphism from R to $\hat{R}$.
   c) There is a map $g : \hat{R} \to \mathbb{Z}$ defined by
$$g(r, n) = n.$$
   What is the kernel of g? Conclude that every rng is an ideal in a ring.
   d) (**Harder**) Consider the ring $C_0(\mathbb{R})$. We can consider the unitalization of $C_0(\mathbb{R})$ as an algebra, which means we take $\hat{C}_0(\mathbb{R}) := C_0(\mathbb{R}) \oplus \mathbb{R}$ instead of $\oplus\mathbb{Z}$. Let us write $\bar{\mathbb{R}} = \mathbb{R} \cup \{x_\infty\}$. Define a map $\varphi : \hat{C}_0(\mathbb{R}) \to C(\bar{\mathbb{R}})$ by
$$(f, \alpha) \mapsto \left(x \mapsto \begin{cases} f(x) + \alpha & x \in \mathbb{R} \\ \alpha & x = x_\infty \end{cases} \right)$$
   for $f \in C_0(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Let $C(S^1)$ denote the ring of continuous functions $h : [-\pi/2, \pi/2] \to \mathbb{R}$ such that $h(-\pi/2) = h(\pi/2)$. Prove that $C(\bar{\mathbb{R}})$ is isomorphic to $C(S^1)$ and conclude that the unitalization of the functions vanishing at infinity on $\mathbb{R}$ is equivalent to the continuous functions on the circle, its one-point compactification. (Hint: construct an isomorphism between functions on $\bar{\mathbb{R}}$ and functions on the interval $[-\pi/2, \pi/2]$ using $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$. Check that under the above constructions, if $f \in C_0(\mathbb{R})$ then the induced function $h \in C(S^1)$ is

continuous on $[-\pi/2, \pi/2]$. Show that this map is injective and surjective, and is a ring homomorphism.)

The above is called the *unitalization of a ring*. The process in (d) above is a method to construct the one-point compactification of a space using ring theory.

3) Even if a ring R has a multiplicative identity 1_R, not every element r of R needs to have a multiplicative inverse r^{-1}. A *multiplicative set* $S \subset R$ is a subset of R (not necessarily a subgroup or subring!) which is closed under multiplication and contains 1_R.
   a) Define a set $S^{-1}R$ as pairs $(s, r) \in S \times R$ modulo the equivalence relation
   $$(s, r) \sim (s', r')$$
   if and only if there exists a $t \in S$ such that $t(sr' - s'r) = 0$. Define addition and multiplication in $S^{-1}R$ as
   $$[s, r] + [s', r'] := [ss', rs' + r's]$$
   $$[s, r] \cdot [s', r'] := [ss', rr'].$$
   Prove that these operations are well-defined and that $S^{-1}R$ forms a ring.
   b) Define a map $f : R \to S^{-1}R$ by $f(r) = (1, r)$. Prove that f is a ring homomorphism. Prove that f is injective if and only if S does not contain any zero divisors of R.
   c) Prove that if $s \in S$, then $f(s)$ has a multiplicative inverse in $S^{-1}R$.
   d) Suppose $0 \in S$. What is $S^{-1}R$?
   e) Suppose R is an integral domain and $S = R \backslash \{0\}$. Prove that $S^{-1}R$ is a field.

The construction above is called the *localization of R at S*, and elements (s, r) are written $\frac{r}{s}$. The localization $S^{-1}R$ adds in a multiplicative inverse for every element of S. The localization $(R\backslash\{0\})^{-1}R$ is called the *field of fraction* of R.