# GROUPS AND RINGS - Unseen Problem Sheet

December 2, 2021

## Rings of fractions

Throughout this Problem Sheet $R$ is a commutative ring. You know that if $a, b, c \in R$, $a$ is not zero nor a zero divisor and $ab = ac$ in $R$, then $b = c$. Thus a non-zero element that is not a zero divisor enjoys some of the properties of a unit without necessarily possessing a multiplicative inverse in $R$. On the other hand, you know that a zero divisor $a$ cannot be a unit in $R$, and if $a$ is a zero divisor we cannot always cancel the $a$'s in the equation $ab = ac$ to obtain $b = c$. The aim of this Problem Sheet is to prove that a commutative ring $R$ is always a subring of a larger ring $Q$ in which every non-zero element of $R$ that is not a zero divisor is a unit in $Q$. The principal application of this will be to integral domains, in which case this ring $Q$ will be a field, called its *field of fractions* or *quotient field*. Indeed, the paradigm for the construction of $Q$ from $R$ is the one offered by the construction of the field of rational numbers from the integral domain $\mathbb{Z}$.

In order to see the essential features of the construction of the field $\mathbb{Q}$ from the integral domain $\mathbb{Z}$, we review the basic properties of fractions. Each rational number may be represented in many different ways as the quotient of two integers (for example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$, etc.). These representations are related by

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc.$$

In more precise terms, the fraction $\frac{a}{b}$ is the equivalence class of ordered pairs $(a, b)$ of integers with $b \neq 0$ under the equivalence relation: $(a, b) \sim (c, d)$ if and only if $ad = bc$. The arithmetic operations on fractions are given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

These are well defined (independent of choice of representatives of the equivalence classes) and make the set of fractions into a commutative ring (in fact, a field), $\mathbb{Q}$. The integers $\mathbb{Z}$ are identified with the subring $\left\{\frac{a}{1} \mid a \in \mathbb{Z}\right\}$ of $\mathbb{Q}$ and every non-zero integer $a$ has an inverse $\frac{1}{a}$ in $\mathbb{Q}$.

It seems reasonable to attempt to follow the same steps for any commutative ring $R$, allowing arbitrary denominators. If, however, $b$ is zero or a zero divisor in $R$, say $bd = 0$, and if we allow $b$ as a denominator, then we should expect to have

$$d = \frac{d}{1} = \frac{bd}{b} = \frac{0}{b} = 0$$

in the "ring of fractions". Thus if we allow zero or zero divisors as denominators, there must be some collapsing in the sense that we cannot expect $R$ to appear naturally as a subring of this "ring of fractions". A second restriction is more obviously imposed by the laws of addition and multiplication: if ring elements $b$ and $d$ are allowed as denominators, then $bd$ must also be a denominator, i.e. the set of denominators must be closed under multiplication in $R$. You will show that these two restrictions are sufficient to construct a ring of fractions for $R$. Note that the following theorem includes the construction of $\mathbb{Q}$ from $\mathbb{Z}$ as a special case.

**Theorem 1.** *Let $R$ be a commutative ring. Let $D$ be any non-empty subset of $R$ that does not contain 0, does not contain any zero divisors and is closed under multiplication (i.e., $ab \in D$ for all $a, b \in D$). Then there is a commutative ring $Q$ such that $Q$ contains $R$ as a subring and every element of $D$ is a unit in $Q$. The ring $Q$ has the following additional properties:*

*(1) every element of $Q$ is of the form $rd^{-1}$ for some $r \in R$ and $d \in D$. In particular, if $D = R \setminus \{0\}$, then $Q$ is a field.*

*(2) (uniqueness of $Q$) The ring $Q$ is the "smallest" ring containing $R$ in which all elements of $D$ become units, in the following sense. Let $S$ be any commutative ring and let $\varphi\colon R \longrightarrow S$ be any injective ring homomorphism such that $\varphi(d)$ is a unit in $S$ for every $d \in D$. Then there is an injective homomorphism $\Phi\colon Q \longrightarrow S$ such that $\Phi|_R = \varphi$. In other words, any ring containing an isomorphic copy of $R$ in which all the elements of $D$ become units must also contain an isomorphic copy of $Q$.*

**Exercise 1.** *In order to prove Theorem 1, let*

$$\mathcal{F} = \{(r, d) \mid r \in R, d \in D\}$$

*and define the relation $\sim$ on $\mathcal{F}$ by*

$$(r, d) \sim (s, e) \qquad \text{if and only if} \qquad re = sd.$$

*(a) Check that $\sim$ is an equivalence relation.*

*(b) Denote the equivalence class of $(r, d)$ by $\frac{r}{d}$:*

$$\frac{r}{d} = \{(a, b) \mid a \in R, b \in D \quad \text{and} \quad rb = ad\} .$$

*Let $Q$ be the set of equivalence classes under $\sim$, and define an additive and multiplicative structure on $Q$:*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

*Prove that $Q$ is a commutative ring.*

*(c) Embed $R$ into $Q$ by defining an injective ring homomorphism*

$$\iota \colon R \longrightarrow Q.$$

*(d) Check that each $d \in D$ has a multiplicative inverse in $Q$.*

*(e) Establish the uniqueness property of $Q$.*

**Definition 1.** *Let $R$, $D$ and $Q$ be as in Theorem 1.*

*(1) The ring $Q$ is called the* ring of fractions *of $R$ with respect to $D$ and is denoted $D^{-1}R$.*

*(2) If $R$ is an integral domain and $D = R \setminus \{0\}$, then $Q$ is called the* field of fractions *or* quotient field *of $R$.*

We now generalise the construction of "ring of fractions" given above by allowing $D$ to contain zero or zero divisors, and so in this case $R$ need not embed as a subring of $D^{-1}R$.

**Theorem 2.** *Let $R$ be a commutative ring and let $D$ be a multiplicatively closed subset of $R$ containing 1. Then there is a commutative ring $D^{-1}R$ and a ring homomorphism $\pi \colon R \longrightarrow D^{-1}R$ satisfying the following universal property: for any homomorphism $\psi \colon R \longrightarrow S$ of commutative rings (that sends 1 to 1) such that $\psi(d)$ is a unit in $S$ for every $d \in D$, there is a unique homomorphism $\Psi \colon D^{-1}R \longrightarrow S$ such that $\Psi \circ \pi = \psi$*

**Exercise 2.** *The proof of Theorem 2 is similar to the one of Theorem 1.*

*(i) How would you define the relation on $R \times D$ in this case? And how would you define $\pi$?*

*(ii) Find* $\ker(\pi)$ *and deduce that* $\pi$ *is an injection if and only if* $D$ *contains no zero divisors or zero of* $R$*.*

*(iii) Prove that* $D^{-1}R = 0$ *if and only if* $0 \in D$*, hence if and only if* $D$ *contains nilpotent elements.*

**Definition 2.** *The ring* $D^{-1}R$ *is called the* ring of fractions *of* $R$ *with respect to* $D$ *or the* localisation of $R$ at $D$.

**Exercise 3.** *Check that the set* $D$ *of invertible elements of a ring* $R$ *satisfies the conditions of Theorem 2, and describe the ring* $D^{-1}R$*.*

**Exercise 4.** *Check that for every ideal* $I$ *of* $R$*, the set* $D = 1 + I$ *satisfies the conditions of Theorem 2.*

**Exercise 5.** *Let* $R = \mathbb{Z}_{12}$ *be the ring of integers mod 12 and let* $D = \{1, 4, 7, 10\} \subseteq R$*. Check that* $D$ *satisfies the conditions of Theorem 2 and that the homomorphism* $\pi\colon R \longrightarrow D^{-1}R$ *is not injective.*