GROUPS AND RINGS
RINGS UNSEEN PROBLEM SHEET 3

1) Let A be a ring with $x^2 = x$ for all $x \in A$. Prove that A is commutative

2) Let A be a commutative ring. Suppose A has a single maximal ideal. Prove the only elements with $x^2 = x$ are 0 and 1.

3) Let A be a commutative ring. **Fact** : Let $x \in A$ be a non-unit. Then x is in some maximal ideal.

Define $\mathcal{J}(A) := \cap_{\mathfrak{m}} \mathfrak{m}$ to the intersection of all maximal ideals. Prove:

$$x \in \mathcal{J}(A) \Leftrightarrow 1 - ax \text{ is not a unit } \forall a \in A$$

4) Show that $\mathbb{Q}$ contains infinitely many integral domains.

5) Let $\mathbb{Z}[i]$ denote the complex numbers of the form $a + bi$ with $a, b \in \mathbb{Z}$.

(1) Show $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$
(2) if p is a prime number, show that

$$\mathbb{Z}[i]/(p) \cong \mathbb{F}_p[x]/(x^2+1)$$

(3) Deduce the ideal (p) is prime if and only if there exists no element $x \in \mathbb{F}_p$ with $x^2 + 1 = 0$
(4) Show that this is the case if and only if $p \equiv 3 \bmod 4$