LINEAR ALGEBRA MATH 50003 Solutions to Problem Sheet 1

1. (a) This map is $T(v) = Av$, where $A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & -3 & 5 \end{pmatrix}$.

Characteristic poly is $(x+1)^2(x-2)$, so eigenvalues are $-1, 2$ with alg multiplicities 2,1 respectively. Geometric multiplicity of the evalue -1 is dimension of the -1 eigenspace, which is 1; geometric mult of 2 is also 1. Since $g(-1) < a(-1)$, there is no basis of evectors.

(b) Matrix of T with respect to the usual basis $1, x, x^2, x^3$ is $\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Characteristic poly is $(x-1)^4$, evalue 1, $a(1) = 4$, $g(1) = 2$. Not diagonalisable.

(c) Matrix of T w.r.t. basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$, which has char poly $(x-3)^2(x-2)^2$. Mults are $a(3) = g(3) = 2$, $a(2) = g(2) = 2$. It is diagonalisable.

(d) T sends $1 \to 0$, $x \to 3x$, $x^2 \to x + 6x^2$, so matrix of T wrt basis $1, x, x^2$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{pmatrix}$. This has distinct evalues 0,3,6, all with alg and geom mult 1, and there is a basis of evectors.

(e) The char poly is $(x+1)^2(x-1)$. The -1 eigenspace consists of solutions to the system $\begin{pmatrix} 0 & a & b \\ 0 & 2 & c \\ 0 & 0 & 0 \end{pmatrix} x = 0$, so it is 2-dimensional iff $ac - 2b = 0$.

2. (a) $|A| = 7$, so A is invertible iff $p \neq 7$. Char poly is $x^2 - 8x + 7 = (x-1)(x-7)$, so evalues are $1, 7$. If $p \neq 2, 3$ these are distinct, so A is diagonalisable by 2.6 in lecture notes. If $p = 2$ or 3, the only evalue is 1, and find that eigenspace E_1 is 1-dimensional, so A is not diagonalisable.

(b) Evalues of B are $1, 2, \alpha$. If $\alpha = 0$ these are distinct, so B is diagonalisable. If $\alpha = 1$, the repeated evalue is 1, and we check that eigenspace E_1 has dim 2, so B is diagble. If $\alpha = 2$, repeated evalue is 2, and find that eigenspace E_2 has dim 1, so B is not diagble.

(c) Characteristic poly of this matrix is $x^2 + 1$. Suppose $p \equiv 3 \bmod 4$. If $\alpha \in \mathbb{F}_p$ is a root of x^2+1, then $\alpha^2 = -1$ and so α is an element of order 4 in the group $\mathbb{F}_p^*$. However $|\mathbb{F}_p^*| = p-1$ is not divisible by 4, so this is a contradiction by Lagrange's theorem. Hence C is not diagonalisable.

Other primes: if $p = 2$ the poly $x^2 + 1 = (x+1)^2$ only has the root 1, and the eigenspace E_1 has dim 1, so C is not diagonalisable. Finally, consider $p \equiv 1 \bmod 4$. It is a famous fact (look it up on the web or in a book!) that the poly $x^2 + 1$ has two distinct roots $\pm\lambda \in \mathbb{F}_p$ in this case, so C is diagonalisable.

3. (a) (1) First, by 1st year linear algebra, $A \sim_1 B$ iff $\exists$ elementary matrices $E_1, \ldots, E_k$ such that $B = E_1 \cdots E_k A$. Obviously $A \sim_1 A$.

So $A \sim_1 B \Rightarrow B = E_1 \cdots E_k A \Rightarrow A = E_k^{-1} \cdots E_1^{-1} B \Rightarrow B \sim_1 A$.

And $A \sim_1 B, B \sim_1 C \Rightarrow B = E_1 \cdots E_k A, C = F_1 \cdots F_l A \Rightarrow C = F_1 \cdots F_l E_1 \cdots E_k A \Rightarrow A \sim_1 C$.

Hence $\sim_1$ is an equivalence relation.

(2) Next, $A \sim_2 B$ if $\exists P$ such that $B = P^{-1}AP$.

Then $A \sim_2 A$ as $A = I^{-1}AI$.

And $A \sim_2 B \Rightarrow B = P^{-1}AP \Rightarrow A = PBP^{-1} \Rightarrow B \sim_2 A$.

Finally $A \sim_2 B$, $B \sim_2 C \Rightarrow B = P^{-1}AP$, $C = Q^{-1}BQ \Rightarrow C = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ) \Rightarrow A \sim_2 C$.

Hence $\sim_2$ is an equivalence relation.

(b) Neither is contained in the other. Eg. the matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are similar but not row-equivalent; and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ are row-equiv but not similar.

4. We'll use the permutation-style definition of the determinant from the 1st year course: this is

$$\det(A) = \sum_{\pi \in S_n} sgn(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}.$$

Now let $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ as in the question.

Consider a term $sgn(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}$ in $det(A)$, where $\pi \in S_n$. Because of the $t \times s$ zero matrix in the bottom left of A, for this term to be non-zero, it is necessary that π sends $\{1, \dots s\} \to \{1, \dots s\}$ and $\{s+1, \dots s+t\} \to \{s+1, \dots s+t\}$. We can write such a π as a product $\pi_1 \pi_2$, where π_1 is a permutation of $\{1, \dots s\}$ and π_2 is a permutation of $\{s+1, \dots s+t\}$. Also $sgn(\pi) = sgn(\pi_1)\, sgn(\pi_2)$. Hence

$$\det(A) = \sum_{\pi_1, \pi_2} sgn(\pi_1) sgn(\pi_2) b_{1,\pi_1(1)} \cdots b_{s,\pi_1(s)}\, d_{s+1,\pi_2(s+1)} \cdots d_{s+t,\pi_2(s+t)} =$$

$$\sum_{\pi_1} sgn(\pi_1) b_{1,\pi_1(1)} \cdots b_{s,\pi_1(s)} \sum_{\pi_2} sgn(\pi_2) d_{s+1,\pi_2(s+1)} \cdots d_{s+t,\pi_2(s+t)} = \det(B)\det(D).$$

5. (a) We are given that A and B are similar, so $\exists\ P$ such that $B = P^{-1}AP$.

1. Then $\det B = (\det P)^{-1} \det A \det P = \det A$.
2. The char poly $c_B(x) = \det(xI - B) = \det P^{-1}(xI - A)P = \det(xI - A) = c_A(x)$.
3. As A, B have the same char poly, they have the same evalues.
4. We have $AP = PB$ and P invertible. Hence

$$v \in \ker(B) \Leftrightarrow PBv = 0 \Leftrightarrow APv = 0 \Leftrightarrow Pv \in \ker(A).$$

Hence $P(\ker(B)) = \ker(A)$, and so $\ker(A)$ and $\ker(B)$ have the same dimension, ie. A and B have the same nullity.

5. Let λ be an evalue of A (also of B). The geom mult $g_A(\lambda) = \dim \ker(A - \lambda I)$. Since $A - \lambda I$ is similar to $B - \lambda I$, $g_B(\lambda) = \dim\ker(B - \lambda I) = \dim\ker(A - \lambda I) = g_A(\lambda)$.

6. A and B have the same rank by the Rank-Nullity theorem and part 4.

7. In the characteristic poly $c_A(x)$, the coefficient of x^{n-1} is $-\mathrm{tr}(A)$ (use the defn of the determinant in evaluating $c_A(x) = \det(xI - A)$). As $c_A(x) = c_B(x)$, it follows that $\mathrm{tr}(A) = \mathrm{tr}(B)$.

8. For any positive integer k, $B^k = (P^{-1}AP)(P^{-1}AP) \cdots (P^{-1}AP) = P^{-1}A^kP$, and similarly, for any polynomial $p(x)$, we have $p(B) = P^{-1}p(A)P$. So $p(A) \sim p(B)$.

(b) Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then A, B share all the quantities listed in (a), but $A \not\sim B$ as $A^2 \neq 0, B^2 = 0$.

6. (a) For any $\alpha, \beta, \gamma \in F \setminus 0$, let $T_{\alpha\beta\gamma}$ be the diagonal matrix with diagonal entries α, β, γ. Check that

$$T_{\alpha\beta\gamma}\, M(a, b)\, T_{\alpha\beta\gamma}^{-1} = M(\beta\alpha^{-1}a,\ \gamma\alpha^{-1}b).$$

Hence the assertion of the question.

Observe that $N - I$ has rank 2, while $M(a, b) - I$ has rank 1. Hence $N \not\sim M(a, b)$.

7. Let $p(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_0$, so that

$$C(p(x)) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{r-1} \end{pmatrix}.$$

We show that this has characteristic poly $p(x)$. The proof goes by induction on n. The char poly is

$$c(x) = det \begin{pmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & -1 & x + a_{n-1} \end{pmatrix}$$

Expand by the first row. By induction the det of the 11-minor is $x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$, so we get

$$c(x) = x\,(x^{n-1}+a_{n-1}x^{n-2}+\cdots+a_1)+(-1)^{n-1}a_0.(-1)^{n-1} = x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0 = p(x).$$

Hence the result by induction.