LINEAR ALGEBRA MATH 50003 Problem Sheet 1

1. For each of the following linear maps $T : V \to V$, find the eigenvalues, and for each eigenvalue λ find its algebraic and geometric multiplicities, and determine whether T is diagonalisable.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(b) V is the vector space of polynomials of degree at most 3 over $\mathbb{R}$, and $T(p(x)) = p(1+x) - p'(1-x)$ for all $p(x) \in V$.

(c) V is the vector space of all 2×2 matrices over $\mathbb{R}$, and $T(A) = MA$ for all $A \in V$, where $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

(d) V the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x+1) - p(x) - p(x-1))$ for all $p(x) \in V$.

(e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(v) = Av$, where $A = \begin{pmatrix} -1 & a & b \\ 0 & 1 & c \\ 0 & 0 & -1 \end{pmatrix}$. (Answer will depend on a, b, c.)

2. For a prime p, let $\mathbb{F}_p$ denote the field of p elements consisting of the numbers $0, 1, \ldots, p-1$ with addition and multiplication defined modulo p, and let $M_n(\mathbb{F}_p)$ denote the set of $n \times n$ matrices over $\mathbb{F}_p$.

(a) Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 6 \end{pmatrix} \in M_2(\mathbb{F}_p)$. For which primes p is A invertible? For which p is A diagonalisable over $\mathbb{F}_p$?

(b) Let $p = 3$ and let $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \alpha \end{pmatrix} \in M_3(\mathbb{F}_3)$. For which values of $\alpha \in \mathbb{F}_3$ is B diagonalisable over $\mathbb{F}_3$?

(c) Let $C = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \in M_2(\mathbb{F}_p)$. Show that if $p \equiv 3 \bmod 4$ then C is not diagonalisable over $\mathbb{F}_p$. What about other primes p?

3. For $n \times n$ matrices A, B, write $A \sim_1 B$ to mean that B can be obtained from A by a sequence of elementary row operations; and $A \sim_2 B$ to mean that B is similar to A (i.e. $\exists\, P$ such that $B = P^{-1}AP$).

(a) Prove that $\sim_1$ and $\sim_2$ are both equivalence relations.

(b) Is either of these relations contained in the other? (i.e. does $A \sim_1 B \Rightarrow A \sim_2 B$, or vice versa?)

4. Let $A = \begin{pmatrix} B & C \\ \mathbf{0} & D \end{pmatrix}$, where B is $s \times s$, D is $t \times t$, C is $s \times t$, and $\mathbf{0}$ is the $t \times s$ zero matrix. Prove that $\det(A) = \det(B)\det(D)$.

5. (a) Let A and B be similar $n \times n$ matrices over a field F. Prove A and B have the same determinant; the same characteristic polynomial; the same eigenvalues; the same nullity; the same geometric multiplicities; the same rank; the same trace. Prove also that for any polynomial $p(x)$, the matrices $p(A)$ and $p(B)$ are similar.

(b) Give an example of two matrices C, D that share all the quantities listed in part (a), but which are not similar.

6. (a) Let F be a field, and for $a, b \in F$ let $M(a, b)$ be the matrix

$$M(a,b) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}.$$

Prove that for any $a, b, c, d \in F \setminus \{0\}$, the matrices $M(a, b)$ and $M(c, d)$ are similar.

(b) Let $N = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Prove that N is not similar to $M(a, b)$ for any $a, b \in F$.

7. Let $p(x)$ be a monic polynomial of degree r over a field F, and let $C(p(x))$ be its $r \times r$ companion matrix, as defined in the lectures. Prove that the characteristic polynomial of $C(p(x))$ is $p(x)$.