Linear Algebra MATH 50003
Solutions to Problem Sheet 10

1. To simplify notation, for $v \in V$ denote by $[v]$ the column vector $[v]_B$. We know from lectures that $(u, v) = [u]^T A[v]$.

Suppose A is not invertible. Then A has 0 as an evalue, so there exists nonzero $w \in V$ such that $A[w] = 0$. Then $(v, w) = [v]^T A[w] = 0$ for all v, so $w \in V^\perp$. Hence $(\, , \,)$ is degenerate.

Now suppose $(\, , \,)$ is degenerate, so there exists a nonzero vector $w \in V^\perp$. Then for all $v \in V$ we have $(v, w) = [v]^T A[w] = 0$. Taking $[v]$ to be standard basis vectors, we see that this forces $A[w] = 0$, hence A is not invertible.

2. (i) This is an inner product (Chapter 14), so is symmetric bilinear and non-degenerate.

(ii) This is not bilinear (eg. $(f_1 + f_2, g) \neq (f_1, g) + (f_2, g)$).

(iii) This is symmetric bilinear. It is degenerate, since $V^\perp = \{f \in V : f(1) = 0\}$.

(iv) This is skew-symmetric bilinear. It is non-degenerate: work out the matrix of $(\, , \,)$ wrt the standard basis $1, x, x^2, x^3$ - this is

$$\begin{pmatrix} 0 & 2 & 2 & 3 \\ -2 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{pmatrix}.$$

Check this is invertible.

3. (i) The form $(A, B) = \text{tr}(AB)$ is bilinear, and is symmetric as $\text{tr}(AB) = \text{tr}(BA)$. To show it is non-degenerate, for $1 \leq i, j \leq 2$, let E_{ij} be the matrix with 1 in the ij-entry and 0 elsewhere. If $A \in V^\perp$, then $\text{tr}(AE_{ij}) = 0$ for all i, j, and it is easy to see from this that $A = 0$.

(ii) An orthogonal basis is v_1, v_2, v_3, v_4 where $v_1 = E_{11}$, $v_2 = E_{22}$, $v_3 = E_{12} + E_{21}$, $v_4 = E_{12} - E_{21}$.

(iii) When $\text{char}(F) = 2$ there is an orthonormal basis, namely $E_{11} + E_{21}$, $E_{11} + E_{12} + E_{21}$, $E_{11} + E_{12}$, E_{22}.

4. Let A be invertible and skew-symmetric over $\mathbb{R}$. By Cor 16.5 of lectures, there is an invertible real matrix P such that $P^T AP = J_m$, a block-diagonal sum of 2×2 matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Taking determinants, $\det(P)^2 \det(A) = \det(J_m) = 1$. Hence $\det(A) = 1/\det(P)^2 > 0$.

5. (i) $|A| = -5$, so A is invertible provided $p \neq 5$.

(ii) $(e_1, e_1) = 1$, so take e_1 as the first basis vector. Next, $e_1^T Ay = y_1 + 2y_2 - 3y_3$, so $e_1^\perp = \{y \in V : y_1 + 2y_2 - 3y_3 = 0\}$. This contains $v_2 = 2e_1 - e_2$, and $(v_2, v_2) = 1$. Finally, $v_2^T Ay = -y_2 - 2y_3$. Hence $(\text{Sp}(e_1, v_2))^\perp$ contains $v_3 = 7e_1 - 2e_2 + e_3$, and $(v_3, v_3) = -5$. So e_1, v_2, v_3 is an orthogonal basis.

The matrix P with these columns satisfies $P^T AP = \text{diag}(1, 1, -5)$.

(iii) If $-5 = \alpha^2$, then $(\alpha^{-1} v_3, \alpha^{-1} v_3) = 1$, so $e_1, v_2, \alpha^{-1} v_3$ is an orthonormal basis.

Conversely, if there exists an orthonormal basis, then $\exists Q$ such that $A = Q^T Q$, so taking dets, $-5 = \det(Q)^2$, and so -5 is a square.

6. (i) We have $Q(x) = x^T Ax$, where

$$A = \begin{pmatrix} 1 & \frac{3}{2} & -\frac{1}{2} \\ \frac{3}{2} & 1 & 3 \\ -\frac{1}{2} & 3 & -1 \end{pmatrix}.$$

As in the previous question we find an orthogonal basis for the corresponding symmetric bilnear form $(x, y) = x^T Ay$. Here is one: v_1, v_2, v_3, where $v_1 = e_1$, $v_2 = 3e_1 - 2e_2$, $v_3 = -4e_1 + 3e_2 + e_3$. Since $Q((v_1) = 1$, $Q(v_2) = -5$, $Q(v_3) = 10$, Q is equivalent to Q' where

$$Q'(x) = x_1^2 - 5x_2^2 + 10x_3^2.$$

(ii) As $Q \sim Q'$ they take the same values in $\mathbb{Q}$. Clearly $Q'(x) = 1$ has a solution $x = (1, 0, 0)$, and $Q'(x) = -1$ has a solution $x = (\frac{1}{2}, \frac{1}{2}, 0)$.

(iii) This is tricky. The answer is no. Here is an argument.

Suppose $x \neq 0$ is a solution of $Q'(x) = 0$. Then clearing denominators, there are integers a, b, c such that $a^2 - 5b^2 + 10c^2 = 0$ (and not all of a, b, c are 0). Then 5 divides a, say $a = 5d$, so we get

$$5d^2 - b^2 + 2c^2 = 0.$$

Hence $b^2 = 5d^2 + 2c^2$ (where $b, c, d \in \mathbb{Z}$). If b, c, d have a common factor greater than 1, we can divide through by the square of this; so we may assume that b, c, d have no common factor greater than 1.

Consider congruences modulo 8. Any square is 0,1 or 4 mod 8. If b is odd then d is odd, so modulo 8 we get $b^2 \equiv 1$, whereas $5d^2 + 2c^2 \equiv 5 + 2k$ where $k = 0, 1$ or 4. This is impossible.

If b is even then d is even, and since b, c, d have no common factor, c is odd. Then modulo 8, we have $b^2 \equiv 0$ or 4, whereas $5d^2 + 2c^2 \equiv (0 \text{ or } 4) + 2$, again a contradiction.

7. We are given $v \neq 0$ such that $Q(v) = 0$. Let $(\, , \,)$ be the corresponding symmetric bilinear form. Since $(\, , \,)$ is non-degenerate, there exists $w \in V$ such that $(v, w) \neq 0$, say $(v, w) = \lambda$. Then for $\alpha \in F$ we have $Q(\alpha v + w) = (\alpha v + w, \alpha v + w) = 2\alpha\lambda + Q(w, w)$. Since $\lambda \neq 0$, this takes all values in F as α varies over F.

8. (i) Let $S = \{\alpha^2 : \alpha \in F^\times\}$. This is easily seen to be a subgroup of $F^\times$. Consider the map $\phi : F^\times \to S$ sending $\alpha \to \alpha^2$. This is a homomorphism with kernel $\{\alpha : \alpha^2 = 1\} = \{\pm 1\}$ of order 2. Hence $|S| = |\text{Im}(\phi)| = |F^\times|/|\ker(\phi)| = \frac{1}{2}(p-1)$.

(ii) Let $X = \{ax^2 : x \in F\}$, $Y = \{-by^2 + c : y \in F\}$. By (i), X and Y both have size $\frac{1}{2}(p-1) + 1$ (the extra 1 for the zero element of F). So X and Y both have size more than $\frac{1}{2}|F|$, and hence they intersect in a non-empty set. So there exist $x, y \in F$ such that $ax^2 = -by^2 + c$.

(iii) Let $n = \dim V \geq 3$ and $Q : V \to F$ a non-degenerate quadratic form. By Thm 16.6 of lectures, Q is equivalent to a quadratic form $Q'(x) = ax_1^2 + bx_2^2 + cx_3^2 + \cdots + dx_n^2$, where $a, b, c, \ldots, d \neq 0$. Take $x_3 = 1$ and $x_i = 0$ for $i \geq 4$. By (ii) there exist values of x_1, x_2 such that $ax_1^2 + bx_2^2 + c = 0$. Then $Q(x) = 0$.

(iv) Let $\alpha \in F^\times \setminus S$, a non-square. Then for the quadratic form $Q(x) = x_1^2 - \alpha x_2^2$, there are no nonzero vectors x for which $Q(x) = 0$.