Linear Algebra MATH 50003 Problem Sheet 10

1. Let V be a finite-dimensional vector space over a field F, and let $(\ ,\ )$ be a symmetric or skew-symmetric bilinear form on V. Let B be a basis of V, and let A be the matrix of $(\ ,\ )$ with respect to B. Prove that $(\ ,\ )$ is non-degenerate if and only if A is invertible.

2. Let V be the vector space over $\mathbb{R}$ consisting of polynomials in x of degree at most 3. Which of the following are bilinear forms on V? Which of these are symmetric, skew-symmetric or non-degenerate?

(i) $(f,g) = \int_0^1 f(x)g(x)\,dx$ for all $f,g \in V$

(ii) $(f,g) = f(1) + g(1)$ for all $f,g \in V$

(iii) $(f,g) = f(1)g(1)$ for all $f,g \in V$

(iv) $(f,g) = f(1)g'(1) - f'(1)g(1) + f(0)g'(0) - f'(0)g(0)$ for all $f,g \in V$.

3. Let $V = M_2(F)$ be the vector space of all 2×2 matrices over a field F, and for $A, B \in V$ define

$$(A,B) = \mathrm{tr}\,(AB),$$

the trace of AB.

(i) Show that $(\ ,\ )$ is a non-degenerate symmetric bilinear form on V.

(ii) Assuming that $\mathrm{char}(F) \neq 2$, find an orthogonal basis of V.

(iii) Does V have an orthogonal basis when $\mathrm{char}(F) = 2$?

4. Let A be an invertible skew-symmetric real matrix. Prove that $\det(A) > 0$.

5. Let $p > 2$ be a prime, let $V = (\mathbb{F}_p)^3$, and let $(\ ,\ )$ be the symmetric bilinear form on V defined by $(x,y) = x^T A y$ for $x, y \in V$, where

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{pmatrix}.$$

(i) For which values of p is the form $(\ ,\ )$ non-degenerate?

(ii) Assume $(\ ,\ )$ is non-degenerate. Find an orthogonal basis of V, and find also a matrix P such that $P^T A P$ is diagonal.

(iii) Show that V has an orthonormal basis iff -5 is a nonzero square in $\mathbb{F}_p$ (ie. $\exists \alpha \in \mathbb{F}_p \setminus 0$ such that $\alpha^2 = -5$).

6. Let $V = \mathbb{Q}^3$ and let Q be the quadratic form on V defined by

$$Q(x) = x_1^2 + x_2^2 - x_3^2 + 3x_1x_2 - x_1x_3 + 6x_2x_3 \quad \text{for } x = (x_1, x_2, x_3)^T \in V.$$

(i) Find a quadratic form Q' that is equivalent to Q over $\mathbb{Q}$, of the form $Q'(x) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2$.

(ii) Do the equations $Q(x) = 1$, $Q(x) = -1$ have solutions $x \in \mathbb{Q}^3$?

(iii) Does the equation $Q(x) = 0$ have a nonzero solution $x \in \mathbb{Q}^3$? (*Hint*: clear denominators to get an equation in integer squares, and consider congruences modulo 8.)

7. Let F be a field, V a finite-dimensional vector space over F, and $Q : V \to F$ a non-degenerate quadratic form. Suppose there exists a nonzero vector $v \in V$ such that $Q(v) = 0$. Prove that Q is surjective.

(*Hint*: show $\exists w \in V$ such that $(v,w) \neq 0$ and consider vectors $\alpha v + w$.)

8. Let $p > 2$ be a prime, and let $F = \mathbb{F}_p$. Denote by $F^\times$ the multiplicative group $F \setminus 0$ of order $p-1$.

(i) Show that the set of squares $\{\alpha^2 : \alpha \in F^\times\}$ is a subgroup of $F^\times$ of order $\frac{1}{2}(p-1)$.

(ii) Let $a, b, c \in F^\times$. Show that there is a solution $x, y \in F$ to the equation

$$ax^2 + by^2 = c.$$

(*Hint*: consider the sets $\{ax^2 : x \in F\}$ and $\{-by^2 + c : y \in F\}$ and work out their sizes. What can you deduce from this?)

(iii) Using part (ii), prove that if V is a vector space over F of dimension at least 3, and $Q : V \to F$ is a non-degenerate quadratic form, then there exists a nonzero vector $v \in V$ such that $Q(v) = 0$.

(iv) Give an example to show that the conclusion of (iii) does not necessarily hold if $\dim V = 2$.