LINEAR ALGEBRA MATH 50003 Solutions to Problem Sheet 2

1. (a) $W + v = W + v' \Rightarrow v \in W + v' \Rightarrow \exists w \in W s.t. v = w + v' \Rightarrow v - v' = w \in W$.

Conversely, $v - v' = w \in W \Rightarrow \forall w_0 \in W, w_0 + v = w_0 + w + v'$ and $w_0 + v' = w_0 - w + v \Rightarrow W + v \subseteq W + v'$ and $W + v' \subseteq W + v$.

(b) Linear indep: suppose $\sum_1^r \alpha_i w_i + \sum_1^s \beta_j v_j = 0$. The first sum is in W, so this gives $\sum_1^s \beta_j (W + v_j) = W$ (the zero vector in V/W). Hence $\beta_j = 0$ for all j. Then $\sum_1^r \alpha_i w_i = 0$, so $\alpha_i = 0$ for all i also.

Span: let $v \in V$. Then $W + v \in V/w$, so $\exists \lambda_j$ such that $W + v = \sum_1^s \lambda_j (W + v_j)$. Hence $v = w + \sum_1^s \lambda_j v_j$ for some $w \in W$, so $\exists \mu_i$ such that $v = \sum_1^r \mu_i w_i + \sum_1^s \lambda_j v_j$.

(c) Let X be a subspace of V/W, and define $Y = \{v \in V : W + v \in X\}$. Show Y is a subspace of V containing W. Clearly $X = Y/W$.

2. (i) Observe that $v_1 = e_1 + e_2 + e_3 + e_4$ is an eigenvector ($Av_1 = -v_1$), and $Ae_1 = -e_1 - v_1$. So the 2-dim subspace $W = \mathrm{Sp}(e_1, v_1)$ is T-invariant.

(ii) Take $B_W = \{e_1, v_1\}$ and $V = \{e_1, v_1, e_3, e_4\}$.

(iii) $[T_W]_{B_W} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$, $[\bar{T}]_{\bar{B}} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$.

3. Let $w_1, \dots, w_r$ be a basis of W, and extend to a basis $w_1, \dots, w_r, v_1, \dots, v_s$ of V. Let $X = \mathrm{Sp}(v_1, \dots, v_s)$. Clearly $V = W + X$. Also $W \cap X = 0$, since if $w \in W \cap X$ then $w = \sum_1^r \alpha_i w_i = \sum_1^s \beta_j v_j$, which implies all α_i, β_j are 0 by the linear independence of $w_1, \dots, w_r, v_1, \dots, v_s$. Hence $V = W \oplus X$.

(b) Let $v \in V$. As $V = Y \oplus Z$, we have $v = y + z$ for unique vectors $y \in Y, z \in Z$. Also $y = y_1 + \cdots + y_r$, $z = z_1 + \cdots + z_s$ for unique $y_i \in Y_i, z_j \in Z_j$. Hence $v = y_1 + \cdots + y_r + z_1 + \cdots + z_s$ with uniqueness, so $V = Y_1 \oplus \cdots \oplus Z_s$.

4. (a) Let $U = T(V), W = \ker(T)$. For $v \in V$, $v = T(v) + (v - T(v))$ and the second term is in W as $T(v - T(v)) = T(v) - T^2(v) - 0$. Hence $V = U + W$. Also if $v \in U \cap W$ then $v = T(x)$ for some x, and $0 = T(v) = T^2(x) = T(x) = v$. So $U \cap W = 0$ and $V = U \oplus W$. Both U and W are T-invariant, $T_W = 0$ and for $v = T(x) \in U$, $T(v) = T^2(x) = T(x) = v$, so $T_U = I_U$.

(b) $U = \mathrm{Sp}(e_1 + e_2, e_1 + e_3)$, $W = \mathrm{Sp}(e_1 + e_2 + e_3)$.

5. (i) If $p(x)$ has degree $\leq r$, so does $p'(x)$. Hence V_r is S-invariant.

(ii) Let W be an S-invariant subspace, and assume $W \neq 0$. Let$f(x)$ be a poly of maximal degree in W, and write $f(x) = a_r x^r + \cdots + a_0$, where $a_r \neq 0$. Then $S^r(f(x)) = f^{(r)}(x) = r! a_r \in W$, and hence the constant poly $1 \in W$. Next, $S^{r-1}(f(x)) = r!\, a_r x + (r-1)! a_{r-1} \ni W$, and hence $x \in W$. Continuing like this, we see that $x^2, \dots, x^r \in W$. Hence $W = V_r$.

(iii) For example $T(x) = x$, so $\mathrm{Sp}(x)$ is a T-invariant subspace not equal to V_r.

(iv) If B is the basis $1, x, \dots, x^n$ then $[T]_B = \mathrm{diag}(0, 1, 2, \dots, n-1)$. Thus the eigenspaces of T are all 1-dimensional, and are the subspaces $\mathrm{Sp}(x^i)$ for $i = 0, \dots, n$. If W is a T-invariant subspace, the restriction T_W has char poly dividing that of T (Prop 5.4), so T_W is also diagonalisable and W is a sum of eigenspaces. Hence $W = \mathrm{Sp}(x^{i_1}, \dots, x^{i_r})$ for some $i_1, \dots, i_r$. There are only finitely many choices for the set $\{i_1, \dots, i_r\}$, so there are finitely many T-invariant subspaces.

6. (a) (i) $xI_n - A = (xI_{n_1} - A_1) \oplus \cdots \oplus (xI_{n_r} - A_r)$, and taking determinants gives (i).

(ii) $\dim E_\lambda(A) = n - \mathrm{rank}(A - \lambda I_n) = \sum_1^r (n_i - \mathrm{rank}(A_i - \lambda I_{n_i})) = \sum \dim E_\lambda(A_i)$.

(iii) Let $A = A_1 \oplus A_2$ and let T be the linear map $v \to Av$ for $v \in V = F^n$. So if B is the standard basis $e_1, \dots, e_{n_1}, e_{n_1+1}, \dots, e_n$ then $[T]_B = A$. Changing the order of the basis to $B' = e_{n_1+1}, \dots, e_n, e_1, \dots, e_{n_1}$, we get $[T]_{B'} = A_2 \oplus A_1$. Hence this is similar to A. The case of a general permutation π can be deduced from this by expressing π as a product of transpositions $(i\,j)$.

(b) (i) $v \in \ker f(T) \Rightarrow f(T)(T(v)) = Tf(T)(v) = 0 \Rightarrow T(v) \in \ker f(T)$.

And $v \in f(T)(V) \Rightarrow v = f(T)(w) \Rightarrow T(v) = Tf(T)(w) = f(T)T(w) \in f(T)(V)$.

(ii) If $v \in V$ then $v = v_1 + \cdots + v_r$ with $v_i \in V_i$, so $f(T)(v) = f(T)(v_1) + \cdots + f(T)(v_r)$, and $f(T)(v_i) \in V_i$ as V_i is T-invariant. This expression is unique since $V = V_1 \oplus \cdots \oplus V_r$. Hence result.

7. Using the algorithm for triangularising given in the lecture notes, the following P's work:

$$P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(Many other P's work of course).

8. (a) Suppose T is triangularisable, and let $B = v_1, \ldots, v_n$ be a basis such that $[T]_B$ is upper triangular. For $1 \leq i \leq n$ let $V_i = \mathrm{Sp}(v_1, \ldots, v_i)$. Then $\dim V_i = i$. $V_i \subset V_{i+1}$ and each V_i is T-invariant.

Conversely, suppose $V_1 \subset \cdots \subset V_n$ are T-invariant with $\dim V_i = i$. Let v_1 be a basis of V_1, extend to a basis v_1, v_2 of V_2, and so on, until we have a basis $B = v_1, \ldots, v_n$ of V such that $v_1, \ldots, v_i$ is a basis of V_i for each i. Then $[T]_B$ is upper triangular.

(b) Let A be upper triangular invertible, and for $1 \leq i \leq n$ let $V_i = \mathrm{Sp}(e_1, \ldots, e_i)$. Then $V_1 \subset \cdots \subset V_n$ are A-invariant with $\dim V_i = i$. Since A is invertible, $AV_i = V_i$, and so $V_i = A^{-1}V_i$, so the V_i are also A^{-1}-invariant. Hence as in (a), A^{-1} is also upper triangular.