LINEAR ALGEBRA MATH 50003 Problem Sheet 2

1. Let V be a finite-dimensional vector space, and let W be a subspace of V.

(a) For $v, v' \in V$, show that $W + v = W + v' \Leftrightarrow v - v' \in W$.

(b) Let $\{w_1, \ldots, w_r\}$ be a basis of W, and $\{W + v_1, \ldots, W + v_s\}$ be a basis of V/W. Prove that $\{w_1, \ldots, w_r, v_1, \ldots, v_s\}$ is a basis of V.

(c) Prove that every subspace of the quotient space V/W is of the form Y/W, where Y is a subspace of V containing W.

2. Let $V = \mathbb{R}^4$, and let $T : V \to V$ be the linear map $v \to Av$, where

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Find a 2-dimensional T-invariant subspace W of V.

(ii) Find a basis B_W of W, and a basis B of V containing B_W.

(iii) Compute the matrices $[T_W]_{B_W}$, $[\bar{T}]_{\bar{B}}$ and $[T]_B$ (where as in lectures, $T_W : W \to W$ is the restriction of T to W, and $\bar{T} : V/W \to V/W$ is the quotient map).

3. Let V be a finite-dimensional vector space.

(a) Let W be a subspace of V. Show that there exists another subspace X of V such that $V = W \oplus X$.

(b) Suppose that Y and Z are subspaces of V satisfying the following conditions:

$V = Y \oplus Z$,
$Y = Y_1 \oplus \cdots \oplus Y_r$ for some subspaces Y_i of Y,
$Z = Z_1 \oplus \cdots \oplus Z_s$ for some subspaces Z_i of Z.

Prove that $V = Y_1 \oplus \cdots \oplus Y_r \oplus Z_1 \oplus \cdots \oplus Z_s$. (This is not quite as obvious as it looks – you need to use the definition of a direct sum.)

4. (a) Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear map such that $T^2 = T$. Prove that there are T-invariant subspaces U, W of V such that $V = U \oplus W$, and the restrictions $T_U = I_U$, $T_W = 0$.

(b) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(v) = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} v$ for $v \in \mathbb{R}^3$. Verify that $T^2 = T$, and find subspaces U, W of $\mathbb{R}^3$ as in part (a).

5. Let $n \geq 1$, and let V_n be the vector space over $\mathbb{R}$ consisting of all polynomials in x of degree at most n. Define linear maps $S, T : V_n \to V_n$ by

$$S(p(x)) = p'(x), \quad T(p(x)) = x\,p'(x)$$

for all $p(x) \in V_n$ (where $p'(x)$ denotes the derivative of $p(x)$).

(i) For $r \leq n$, show that V_r is an S-invariant subspace of V_n.

(ii) Let W be an S-invariant subspace of V_n. Prove that $W = V_r$ for some r.

(iii) Find a T-invariant subspace of V_n that is not equal to V_r for any r.

(iv) Prove that there are only finitely many T-invariant subspaces of V_n.

6. (a) Let $A = A_1 \oplus \cdots \oplus A_r$, a block-diagonal matrix (where each A_i is $n_i \times n_i$). Let $c_A(x)$ be the characteristic polynomial of A, and for a scalar λ, let $E_\lambda(A)$ be the λ-eigenspace $\ker(A - \lambda I)$. Prove that

(i) $c_A(x) = \prod_{i=1}^{r} c_{A_i}(x)$,

(ii) for any scalar λ, $\dim E_\lambda(A) = \sum_{i=1}^{r} \dim E_\lambda(A_i)$

(iii) for any permutation π of $\{1, \ldots, r\}$, A is similar to $A_{\pi(1)} \oplus \cdots \oplus A_{\pi(r)}$. (Start by showing that $A_1 \oplus A_2 \sim A_2 \oplus A_1$.)

(b) Let V be finite-dimensional over a field F, and $T : V \to V$ a linear map. Let $f(x) \in F[x]$ be a polynomial.

(i) Prove that the subspaces $\ker(f(T))$ and $f(T)(V)$ (this is the same as $\mathrm{Im}(f(T))$) are both T-invariant.

(ii) Suppose $V = V_1 \oplus \cdots \oplus V_r$, where each V_i is T-invariant. Prove that $f(T)(V) = f(T)(V_1) \oplus \cdots \oplus f(T)(V_r)$.

7. For each of the following matrices A, find an invertible matrix P over $\mathbb{C}$ such that $P^{-1}AP$ is upper triangular:

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

8. (a) Let V be an n-dimensional vector space and $T : V \to V$ a linear map. Say that T is *triangularisable* if there exists a basis B of V such that $[T]_B$ is an upper triangular matrix.

Prove that T is triangularisable if and only if there is a sequence of subspaces $V_1, \ldots, V_n$ such that $V_1 \subset V_2 \subset \cdots \subset V_n$, each V_i is T-invariant, and $\dim V_i = i$ for all i.

(b) Let A be an invertible upper triangular matrix over a field F. Prove that A^{-1} is also upper triangular.