LINEAR ALGEBRA MATH 50003 Problem Sheet 3

1. (a) Find a real 3×3 matrix with characteristic polynomial $x^3 - 2x^2 + 3x - 1$.

(b) Find a real 4×4 invertible matrix A such that $A^{-1} = A^3 + A + I$.

(c) Find a 5×5 invertible matrix A over the field $\mathbb{F}_3$ such that $A^{-1} = 2A^3 + 2I$ and $A \neq I$.

(d) Find a 3×3 matrix A over $\mathbb{F}_2$ such that $A^7 = I$ and $A \neq I$.

(e) For any positive integer n and any field F, find an $(n-1) \times (n-1)$ matrix A over F such that $A^n = I$ and $A \neq I$.

2. Let A be an $n \times n$ matrix over a field F, and suppose $A^k = 0$ for some positive integer k. Prove that $A^n = 0$.

3. Let A be the real matrix $\begin{pmatrix} 1 & -2 & 2 \\ 0 & -2 & 3 \\ 0 & -4 & 5 \end{pmatrix}$.

(a) Find a polynomial $p(x)$ such that $A^{-1} = p(A)$.

(b) Find a polynomial $q(x)$ of degree 2 such that $A^4 = q(A)$.

(c) Find all the eigenvectors of $A^3 - 4A^2$.

4. (a) Give a direct proof of the Cayley-Hamilton Theorem for upper triangular matrices.

(b) Using the Triangularisation Theorem, deduce Cayley-Hamilton for arbitrary $n \times n$ matrices over $\mathbb{C}$.

5. (a) Prove that if A, B are $n \times n$ matrices over any field, then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. (Here, as always, $\mathrm{tr}(A)$ is the trace of A, ie. the sum of the diagonal entries of A.)

(b) Let A, B be 2×2 matrices such that $(AB)^2 = 0$. Prove that $(BA)^2 = 0$. (Hint to get started: apply Cayley-Hamilton for 2×2 matrices as given in lectures to AB.)

(c) Does (b) apply also to 3×3 matrices? Either give a proof or find a counterexample.

6. Let F be a field, and let f, g be nonzero polynomials in $F[x]$.

(a) Prove that $\gcd(f, g)$ is unique, up to scalar multiplication.

(b) Prove that $\mathrm{lcm}(f, g) = \frac{fg}{\gcd(f,g)}$.

7. Let f, g be the following polynomials over $\mathbb{R}$:

$$f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4, \quad g(x) = x^3 + x^2 - 4x - 4.$$

Find $d = \gcd(f, g)$, and find also polynomials r, s such that $d = rf + sg$.

8. (a) Prove that every irreducible polynomial in $\mathbb{R}[x]$ is either linear, or quadratic with no real roots.

(b) Find all the irreducible polynomials in $\mathbb{F}_2[x]$ of degree 4.

(c) Find all the irreducible monic quadratic polynomials in $\mathbb{F}_3[x]$.

(d) Find an irreducible cubic polynomial in $\mathbb{F}_5[x]$.

(e) Factorize $x^4 + 1$ as a product of irreducible polynomials in $\mathbb{F}_2[x]$.

(f) Factorize $x^7 + 1$ as a product of irreducible polynomials in $\mathbb{F}_2[x]$.

9. (a) Let $p(x) \in \mathbb{Q}[x]$ be a monic polynomial with integer coefficients, and let $\alpha \in \mathbb{Q}$ be a root of $p(x)$. Prove that $\alpha \in \mathbb{Z}$.

(b) For which positive integers $k \leq 100$ is $x^3 + x + k$ reducible in $\mathbb{Q}[x]$?

(c) Show that $x^4 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

10. Let A be an $n \times n$ matrix over $\mathbb{C}$, and suppose that the trace $\mathrm{tr}(A^i) = 0$ for all positive integers i. Prove that $A^n = 0$. (Hint to get started: use the Triangularisation Theorem for A.)