Linear Algebra Math 50003 Problem Sheet 4

1. (a) Let $B = P^{-1}AP$. For any polynomial $p(x)$, we have $p(B) = P^{-1}p(A)P$, so $p(A) = 0 \Leftrightarrow p(B) = 0$. So clearly $m_A(x) = m_B(x)$.

(b) Let $m(x) = \text{lcm}(m_{A_1}(x), \dots, m_{A_k}(x))$. Observe that for any poly $p(x)$ we have $p(A) = p(A_1) \oplus \dots \oplus p(A_k)$. As $m_{A_i}|m$ for all i, it follows that $m(A) = m(A_1) \oplus \dots \oplus m(A_k) = 0$. Also if $p(A) = 0$ then $p(A_i) = 0$ for all i, so $m_{A_i}|p$ for all i, hence $m|p$. Therefore $m = m_A$.

(c) The standard basis $e_1, \dots, e_n$ consists of evectors of A, so for each i there exists j such that $Ae_i = \lambda_j e_i$, and hence $\prod_{j=1}^k (A - \lambda_j I)e_i = 0$. Therefore $\prod_{j=1}^k (A - \lambda_j I) = 0$. As each λ_j must be a root of the min pol by 9.2 of lecs, it follows that $m_A(x) = \prod_1^k (x - \lambda_j)$.

(d) Suppose $T : V \to V$ is diagonalisable. Then by Cor 10.2 of lecs, $m_T(x) = \prod_1^k (x - \lambda_j)$, a product of distinct linear factors. If W is a T-invariant subspace, then m_{T_W} divides m_T by 9.4 of lecs, so m_{T_W} is also a product of distinct linear factors. Hence T_W is diagonalisable by 10.2.

2. (a) Let these matrices be A and B. Compute char poly $c_A(x) = (x - 1)^3$ and $(A - I)^2 = 0$, so $m_A(x) = (x - 1)^2$. Also $c_B(x) = x^3(x - 4)$ and $m_B(x) = x(x - 4)$.

(b) For the basis $B = \{1, x, x^2, x^3\}$, matrix $[T]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So char pol $c_T(x) = x^4$. If $F = \mathbb{R}$ then $m_T(x) = x^4$; if $F = \mathbb{F}_2$ it is x^2; and if $F = \mathbb{F}_3$ it is x^3.

3. (a) Let $A = C(p(x))$. We know from Sheet 1 qn that A has char poly $p(x)$, so $m_A(x)|p(x)$. If $e_1, \dots, e_n$ is the standard basis, then this basis is $e_1, Ae_1, \dots, A^{n-1}e_1$. These vectors are linearly indep, which means that there is no nonzero poly $f(x)$ of degree $\leq n - 1$ such that $f(A) = 0$. Hence m_A has degree n and $m_A(x) = p(x)$.

(b) (i) Here $c_A(x) = (x - \lambda)^n$. If $a_1, \dots, a_{n-1} \neq 0$ then $e_n, Ae_n, \dots, A^{n-1}e_n$ is a basis (consisting of nonzero multiplies of the standard basis), so as in (a) we see that $m_A(x)$ has degree n, so is equal to $c_A(x)$.

(ii) If $a_i = 0$, then $A - \lambda I$ is block-diagonal with at least two blocks, so by Q1(b), $m_A(x)$ is the lcm of some powers of $(x - \lambda)$ of degree less than n. Hence $m_A(x) = (x - \lambda)^k$ with $k < n$.

4. (a) if $T^k = 0$ then the min poly of T divides x^k, so it is x^l for some l. Also $l \geq 2$ as $T \neq 0$. Hence $m_T(x)$ is not a product of distinct linear factors, so T is not diagonalisable by 10.2 of lec notes.

(b) Suppose $T^k = I_V$. If $p(x) = x^k - 1$ then $p(T) = 0$, so $m_T(x)$ divides $x^k - 1$. Over $\mathbb{C}$ this factorizes as $\prod_{j=0}^{k-1}(x - \omega^j)$, where $\omega = e^{2\pi i/k}$. Hence $m_T(x)$ is a product of distinct linear factors, so T is diagonalisable by 10.2.

(c) (i) If k is the order of π (as an element of the symmetric group), then $T^k = I$, so T is diagonalisable by (b).

(ii) Let $\omega = e^{2\pi i/n}$, and for $i = 0, \dots n - 1$ let

$$v_i = \sum_{j=0}^{n-1} \omega^{ij} v_j.$$

Then $v_0, \dots, v_{n-1}$ is a basis of evectors.

5. (a) The char poly of T is $(x-2)^2(x-3)^3$, so Primary Decomp is $V = V_1 \oplus V_2$, where $V_1 = \ker(T - 2I)^2$, $V_2 = \ker(T - 3I)^2$. Compute that $V_1 = \text{Sp}(e_2 - e_3 - e_4, e_1 - 7e_3 - e_4)$, $V_2 = \text{Sp}(e_2, e_4)$.

(b) Take $B = \{e_2, e_4, e_2 - e_3 - e_4, e_1 - 7e_3 - e_4\}$.

6. (a) As g_1, g_2 are coprime, there exist $r, s \in F[x]$ such that $rg_1 + sg_2 = 1$. Hence for $v \in V$ we have

$$v = r(T)g_1(T)(v) + s(T)g_2(T)(v).$$

The first vector $r(T)g_1(T)(v) \in \ker g_2(T) = V_2$, so $P_2(v) = r(T)g_1(T)(v)$ and similarly $P_1(v) =$

$s(T)g_2(T)(v)$. Hence $P_2 = r(T)g_1(T)$ and $P_1 = s(T)g_2(T)$.

(b) In Q5(a), $g_1 = (x-2)^2$ and $g_2 = (x-3)^2$. Use Euclidean Alg to compute that $r = -(2x-7)$, $s = 2x - 3$.

7. (a) Note that $(BA)^{k+1} = B(AB)^k A$. Hence if $q(x) = xp(x)$ then $q(BA) = B\,p(AB)\,A$. So if $p(AB) = 0$ then $q(BA) = 0$.

(b) We deduce that $m_{BA}(x)$ divides $xm_{AB}(x)$. Also by symmetry, $m_{AB}(x)$ divides $xm_{BA}(x)$

(c) If A and B are invertible, so is AB, so 0 is not a root of its min poly. Hence by (b), $m_{BA} = m_{AB}$.

(d) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $AB = 0$ has min poly x, whereas $BA = A$ has min poly x^2.