Linear Algebra Math 50003 Problem Sheet 4

1. (a) Prove that if A and B are similar $n \times n$ matrices, then $m_A(x) = m_B(x)$ (where $m_A(x)$ is the minimal polynomial of A).

(b) For $i = 1, \ldots, k$ let A_i be an $n_i \times n_i$ matrix, and let $A = A_1 \oplus \cdots \oplus A_k$. Prove that $m_A(x) = \text{lcm}\,(m_{A_1}(x), \ldots, m_{A_k}(x))$.

(c) Let A be a diagonal matrix with characteristic polynomial $\prod_{i=1}^{k}(x - \lambda_i)^{a_i}$, where $\lambda_1, \ldots, \lambda_k$ are distinct. Prove that $m_A(x) = \prod_{i=1}^{k}(x - \lambda_i)$.

(d) Let V be a finite-dimensional vector space, and suppose $T : V \to V$ is a diagonalisable linear map. Show that if W is a T-invariant subspace of V, then the restriction $T_W : W \to W$ is also diagonalisable.

2. (a) Find the minimal polynomials of the following matrices over $\mathbb{R}$:

$$\begin{pmatrix} -2 & -6 & -9 \\ 3 & 7 & 9 \\ -1 & -2 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(b) Let V be the vector space consisting of all polynomials of degree at most 3 over a field F, and let $T : V \to V$ be the linear map defined by $T(p(x)) = p'(x)$ (the derivative of $p(x)$) for all $p(x) \in V$. Find the minimal polynomial of T in the following cases:

(i) $F = \mathbb{R}$

(ii) $F = \mathbb{F}_2$ (the field of 2 elements)

(iii) $F = \mathbb{F}_3$.

3. (a) Let F be a field, and let $p(x) \in F[x]$ be a monic polynomial. Prove that the minimal polynomial of the companion matrix $C(p(x))$ is $p(x)$.

(b) Let $\lambda, a_1, \ldots, a_{n-1} \in F$, and let A be the $n \times n$ matrix over F that has λ's on the diagonal and $a_1, \ldots, a_{n-1}$ above the diagonal, ie.

$$A = \begin{pmatrix} \lambda & a_1 & & & \\ & \lambda & a_2 & & \\ & & \ddots & & \\ & & & \lambda & a_{n-1} \\ & & & & \lambda \end{pmatrix}.$$

(i) Show that if $a_i \neq 0$ for all i, then $m_A(x) = (x - \lambda)^n$.

(ii) Show that if $\exists i$ such that $a_i = 0$, then $m_A(x) = (x - \lambda)^k$ for some $k < n$.

4. Let V be a vector space of dimension $n \geq 2$ over $\mathbb{C}$, and let $T : V \to V$ be a linear map.

(a) Show that if $T \neq 0$ but $T^k = 0$ for some k, then T is not diagonalisable.

(b) Show that if $T^k = I_V$ for some k, then T is diagonalisable.

(c) Let $\pi \in S_n$ (the symmetric group), let $v_1, \ldots, v_n$ be a basis of V, and define T to be the linear map such that $T(v_i) = v_{\pi(i)}$ for all i.

(i) Show that T is diagonalisable.

(ii) In the case where $\pi = (1\,2\,\cdots\,n) \in S_n$, find a basis of V consisting of eigenvectors of T.

5. Let $V = \mathbb{R}^4$ and let $T : V \to V$ be the linear map $T(v) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 6 & 3 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 3 \end{pmatrix} v$ for $v \in V$.

(a) Find the Primary Decomposition of V with respect to T.

(b) Find a basis B of V such that $[T]_B$ is block-diagonal with 2×2 blocks.

6. Let V be a vector space over F and $T : V \to V$ a linear map with minimal polynomial $m_T(x) = g_1(x)g_2(x)$, where $g_1, g_2 \in F[x]$ are coprime polynomials. Define $V_i = \ker g_i(T)$, so $V = V_1 \oplus V_2$ by

Proposition 10.3 of lectures. For $i = 1, 2$, let $P_i : V \to V_i$ be the *projection* map defined as follows: for $v \in V$, write $v = v_1 + v_2$ for unique $v_i \in V_i$ and define

$$P_i(v) = v_i.$$

(a) For $i = 1, 2$ show that there exist polynomials $p_i(x) \in F[x]$ such that $P_i = p_i(T)$.

(b) Find such polynomials $p_1(x), p_2(x)$ for the Primary Decomposition $V = V_1 \oplus V_2$ in Q5(a).

7. Let A, B be $n \times n$ matrices over a field F.

(a) Suppose $p(x) \in F[x]$ is a polynomial such that $p(AB) = 0$. Show if $q(x) = xp(x)$, then $q(BA) = 0$.

(b) What can be deduced from (a) about the relationship between the minimal polynomials of AB and BA?

(c) Deduce that if A and B are both invertible, then $m_{AB}(x) = m_{BA}(x)$.

(d) Give an example of matrices A, B for which $m_{AB}(x) \neq m_{BA}(x)$