**Linear Algebra MATH 50003**
**Solutions to Problem Sheet 5**

**1.** (i) $J_1(0) \oplus J_1(-1-i)^2 \oplus J_1(3)^3$, $J_1(0) \oplus J_1(-1-i)^2 \oplus J_2(3) \oplus J_1(3)$, $J_1(0) \oplus J_1(-1-i)^2 \oplus J_3(3)$, $J_1(0) \oplus J_2(-1-i) \oplus J_1(3)^3$, $J_1(0) \oplus J_2(-1-i) \oplus J_2(3) \oplus J_1(3)$, $J_1(0) \oplus J_2(-1-i) \oplus J_3(3)$. Phew!

(ii) If $p \neq 3$ then in $\mathbb{F}_p$ we have $1 \neq -2$ so the JCFs have two evalues. There are 3 possible JCFs with char poly $(x-1)^3$ ($J_3(0)$, $J_2(0) \oplus J_1(0)$ etc) and 7 with char poly $(x+2)^5$ ($J_5(-2)$, $J_4(-2) \oplus J_1(-2)$ etc). So there are $3 \times 7 = 21$ JCFs with char poly $(x-1)^3(x+2)^5$.

If $p = 3$ then $1 = -2$ in $\mathbb{F}_p$ and the char poly is $(x-1)^8$. The number of JCFs is the number of partitions of 8 (ie. the number of expressions of the form $8 = n_1 + \cdots + n_k$ where $n_i$ are positive integers), which is 22.

(iii) JCF must have Jordan blocks $J_3(0)$ and $J_2(1)$, plus others of dimension adding to 2 with evalues 0,1: there are 5 such: $J_2(0)$, $J_1(0)^2$, $J_2(1)$, $J_1(1)^2$, $J_1(0) \oplus J_1(1)$.

**2.** Matrix 1: over $\mathbb{C}$ or $\mathbb{F}_p$, $p \neq 2$, this has 3 distinct evalues so is diagonalisable, so JCF is $J_1(1) \oplus J_1(0) \oplus J_1(-1)$. Over $\mathbb{F}_2$, JCF is $J_2(1) \oplus J_1(0)$.

Matrix 2: char pol is $x^3$, and $g(0) = 1$ (for any $F$): so JCF is $J_3(0)$.

Matrix 3: char pol is $(x+1)(x-2)^2$. Over $\mathbb{C}$ or $\mathbb{F}_p, p \neq 3$, JCF is $J_1(-1) \oplus J_2(2)$; over $\mathbb{F}_3$, JCF is $J_2(2) \oplus J_1(2)$.

Matrix 4: char pol $x^5$, min pol $x^4$, $g(0) = 2$, so JCF $J_4(0) \oplus J_1(0)$.

Matrix 5: over $\mathbb{C}$ or $\mathbb{F}_p, p \neq 2$, JCF is $J_3(1) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1)$; over $\mathbb{F}_2$, JCF is $J_3(1) \oplus J_3(1) \oplus J_1(1)$.

**3.** JCFs are respectively

$$J_4(2),\ J_3(2) \oplus J_1(2),\ J_3(2) \oplus J_1(2),\ J_2(2) \oplus J_1(2) \oplus J_1(2),\ J_3(2) \oplus J_1(2).$$

So numbers 2,3 and 5 are similar.

**4.** Evalue is 1; rank $(A-I) = n-3$ implies $g(1) = 3$, so there are 3 Jordan blocks; and rank $(A-I)^{n-4} = 1$ implies there is 1 block of size $n-3$ and no larger ones. Hence JCF is $J_{n-3}(1) \oplus J_2(1) \oplus J_1(1)$.

**5.** (a) Clearly $T^n = I$. Hence $T$ is diagonalisable by Q4 of Sheet 4. The evalues are $n^{th}$ roots of unity in $\mathbb{C}$, and each $n^{th}$ root $\omega$ occurs, since $v_1 + \omega v_2 + \omega^2 v_3 + \cdots + \omega^{n-1} v_n$ is an evector of $T$ with evalue $\omega$. Hence JCF is $J_1(1) \oplus J_1(\omega) \oplus \cdots \oplus J_1(\omega^{n-1})$.

(b) The evalues $\omega^j$ all lie in $\mathbb{R}$ iff $n \leq 2$. So these are the only values for which $T$ has a JCF over $\mathbb{R}$.

(c) Let $F = \mathbb{F}_p$. The min poly $m_T(x)$ divides $x^p - 1$. Over $\mathbb{F}_p$ this factorizes as $(x-1)^p$, so the only evalue of $T$ is 1. From the matrix of $T$ wrt the given basis, we see that rank$(T - I) = n - 1$, so $g(1) = 1$ and the JCF has 1 Jordan block. So JCF of $T$ is $J_p(1)$.

**6.** (a) Let $E$ be the standard basis in order $e_1, \ldots, e_n$ and $F$ the standard basis in reverse order $e_n \ldots, e_1$. As $Je_n = \lambda e_n + e_{n-1}$, $Je_{n-1} = \lambda e_{n-1} + e_{n-2}$, etc, the linear transformation $S(v) = Jv$ satisfies $[S]_E = J$, $[S]_F = J^T$. So if $P$ is the change of basis matrix from $E$ to $F$, $P^{-1}JP = J^T$. Therefore $J$ and $J^T$ are similar.

Relevant for part (c): Note also that $P$ is the matrix with 1's on the "reverse diagonal", and is symmetric.

(b) Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. By the JCF theorem $A$ is similar to a JCF matrix $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. By part (a), for each $i$, $\exists P_i$ such that $P_i^{-1} J_{n_i}(\lambda_i) P_i = J_{n_i}(\lambda_i)^T$. If we let $P$ be the block-diagonal matrix $P_1 \oplus \cdots \oplus P_k$, then $P^{-1} = P_1^{-1} \oplus \cdots \oplus P_k^{-1}$ and so

$$P^{-1}JP = P_1^{-1}J_{n_1}(\lambda_1)P_1 \oplus \cdots \oplus P_k^{-1}J_{n_k}(\lambda_k)P_k = J_{n_1}(\lambda_1)^T \oplus \cdots \oplus J_{n_k}(\lambda_k)^T = J^T.$$

As $A \sim J$, $\exists Q$ such that $Q^{-1}AQ = J$. Then

$$P^{-1}Q^{-1}AQP = P^{-1}JP = J^T = Q^T A^T (Q^{-1})^T.$$

Noting that $(Q^{-1})^T = (Q^T)^{-1}$, this gives

$$(Q^T)^{-1}P^{-1}Q^{-1}\, A\, QPQ^T = A^T,$$

and hence $A \sim A^T$.

(c) As noted in (a), the matrix $P$ above is symmetric. Hence $(QPQ^T)^T = QP^TQ^T = QPQ^T$, and so $QPQ^T$ is symmetric.

**7.** (a) Consider $J_n(\lambda) = J + \lambda I$ where $J = J_n(0)$. Then $J_n(\lambda)^2 = J^2 + 2\lambda J + \lambda^2 I$. This has $\lambda^2$ on the diagonal, $2\lambda$ on the next diagonal up, and 1 on the next diagonal. As $\lambda \neq 0$, we see that $J_n(\lambda)^2 - \lambda^2 I$ has rank $n-1$, which means the geom mult of the evalue $\lambda^2$ is 1. Therefore the JCF of $J_n(\lambda)^2$ is $J_n(\lambda^2)$.

(b) This is a really nice application of the JCF theorem.

First we show that any Jordan block $J_n(\lambda)$ with $\lambda \neq 0$ has a square root. To see this, let $\mu$ be a square root of $\lambda$ in $\mathbb{C}$. By (i), $\exists P$ s.t. $J_n(\lambda) = P^{-1}J_n(\mu)^2P$, and the RHS is equal to $(P^{-1}J_n(\mu)P)^2$.

By the JCF thm, $A$ is similar to a JCF matrix $J = J_1 \oplus \cdots \oplus J_k$, where $J_i = J_{n_i}(\lambda_i)$. As $A$ is invertible, each $\lambda_i \neq 0$, so each $J_i$ has a square root, say $J_i = K_i^2$. Then $J = K^2$, where $K = K_1 \oplus \cdots \oplus K_k$. So $J$ has a square root, and so does $A$, using the argument in the previous para.